Practice exam #3....SOLUTIONS.... Math 110. Fall 1998.

1. (a) Implicitly differentiate each of the following:

$\frac{d}{dx}\left[x^2+y^2+\sin(xy)\right] =$
$ANSWER:\ 2x+2yy'+\cos(xy)(y+xy')$

$\frac{d}{dy}\left[x^2+y^2+\sin(xy)\right] =$
$ANSWER:\ 2xx'+2y+\cos(xy)(x'y+x)$

(b) Find the slope of the curve $y+x\ln(y)=(x+1)\ln(x+1)+1$ at $x=0$.

ANSWER:
Differentiate both sides with respect to x:
$y'+\ln(y)+(x/y)y'=\ln(x+1)+(x+1)/(x+1)+0$

Now you can solve for y'. But before doing so, it will save us time if we first plug in values for x and y.
To find what y is, plug in $x=0$ into the original equation:
$y+0=0+1$ (since $\ln(0+1)=0$), so $y=1$.

So we get: $y'+0+0=0+1$, so $y'=1$. The slope at $x=0$ is 1.

(c) Find the second derivative d^2y/dx^2 for the implicit function $x^2+y^2=1$.

ANSWER:
One way to do this is to first solve explicitly for y, and then differentiate twice. A harder way is to differentiate implicitly, without first solving for y:

$2x+2yy'=0$. So $y'=-x/y$. So $y''=[-x/y]'=\frac{y(-1)-(-x)y'}{y^2}$. Now, we can plug in $y'=-x/y$ into this last expression to obtain y'' only in terms of x and y.

The first way is definitely easier. So why do we bother with the harder way? It's because the first method, although easier, may sometimes not work, whereas the second method, although harder, will always work, no matter how complicted our equation is!

2. A ladder is leaning against a wall. The bottom of the ladder is slipping away from the wall at a constant rate of 2 ft/sec. Does a person standing on top of the ladder fall faster and faster, or slower and slower, or at a constant rate, as the ladder slips? Justify your answer by using the methods of implicit differentiation and related rates of change.

ANSWER:
This is somewhat similar to the fish problem we did in class.
Let $x=$ the distance of the foot of the ladder from the wall.
Let $y=$ the height of the top of the ladder.
Let $l=$ the length of the ladder.
Then we have a right triangle, and by the Pythagorean Theorem, we have

$$x^2+y^2=l^2$$

. Let t denote time. Then, according to the problem, $dx/dt=$ ____ ($\leftarrow$ fill in the blank!)
And $dl/dt=$ ____.

We are looking for dy/dt, the rate at which the height of the ladder is changing. So implicity differentiate the equation above:
$\frac{d}{dt}\,[x^2+y^2] = \frac{d}{dt}\,[l^2]$.
We get: $2x(dx/dt) + 2y(dy/dt) = 0$.
Now plug in 2 for dx/dt, and solve for dy/dt.
After a couple of steps, we eventually get:

$$dy/dt = -2x/y$$

Now here's the part where you have to think.
As time goes on, does x get larger or smaller?
As time goes on, does y get larger or smaller?
So as As time goes on, does the absolute value of dy/dt get larger or smaller?
So does the person fall faster or slower as time goes on?

3. (a) Sketch a graph of the function $f(x) = (x-1)^2 - 3$.

ANSWER:
You can do this with your calculator, but hopefully can also do it without: it's just a parabola $y = x^2$ shifted 1 to the right and 3 down.

(b) Suppose we're trying to find the two roots of this function using Newton's Method. If we start with $x = 2$ as our initial guess for Newton's Method, which of the two roots will we get? Explain why this happens.

ANSWER:
We get closer to the root on the right. This is because the tangent line at $x = 2$ gets us closer to that root.

(c) If we start with $x = 1$ as our initial guess for Newton's Method, which of the two roots will we get? Explain PICTORIALLY why this happens.

ANSWER:
We get neither root. We get a zero in the denominator when we plug into the formula for Newton's Method, because the tangent line at $x = 1$ is horizontal. Pictorially, the tangent line doesn't cross the x-axis, so we don't get a "next guess".

4. Evaluate the following limits. You may use L'Hopital's rule where appropriate. Show all work.

(a) $\lim_{x\to 1} x^2/e^x = 1/e$.

(b) $\lim_{x\to\infty} x^2/e^x$
Use L'Hopital's rule, since both the top and the bottom are going to infinity. So we get:
$\lim_{x\to\infty} 2x/e^x$
Again use L'Hopital's rule, since both the top and the bottom are going to infinity. So we get:
$\lim_{x\to\infty} 2/e^x$
Now we can't use L'Hopital's rule anymore. (Why not?) We just find the limit of this last expression by plugging in large values for x.

(c) $\lim_{x\to-\infty} x^2/e^x = 0$ (plug in -100 for x and see what happens).

(d) $\lim_{x\to 0^+} \ln(x)/e^x = -\infty$ (plug in .01).

(e) $\lim_{x\to 1} \ln(x)/(x-1) = 1$ (use L'Hopital's rule).

(f) $\lim_{x\to 0^+} \ln(x)/\sin(x) = \infty$ (use L'Hopital's rule).

(g) $\lim_{x\to 2} 1/\ln(2-x) =$ DNE (plug in 1.99 AND 2.01, see what you get for each).

(h) $\lim_{x\to 0} \ln(x)/\sin(x) =$ DNE (the limit from the LEFT does not exist; why?).

(i) $\lim_{x\to 0} x/x = 1$ (can either use L'Hopital's rule, or just plug in values close to zero, on the left and right).

(j) $\lim_{x\to 0} x/|x| =$ DNE (plug in values on the left and the right; we get different answers, 1 and -1).

(k) $\lim_{x\to 0} x^{2x}$ CANCELLED.

(l) $\lim_{x\to 0} \cos(x^{2x})$ CANCELLED.

5. (a) Find all horizontal and vertical asymptotes of the function $f(x) = \dfrac{e^x}{x^{1000}}$.

ANSWER:
To find horizontal asymptotes, take the limit of $f(x)$ as x goes to infinity, and to negative infinity. It turns out that there is no horizontal asymptote to the right, but the x axis is a horizontal asymptote on the left.

To find vertical asymptotes, see where the function is undefined. Then take limits from the left and the right; we get a vertical asymptote at $x = 0$.

(b) Find the domain and all critical points of $f(x)$.

ANSWER:
The domain is all real numbers except 0.

$f'(x) = \dfrac{x^{1000}e^x - 1000x^{999}}{x^{2000}}$.
f' is undefined at $x = 0$, but 0 is not in the domain of f, so it does not qualify as a critical point. Set f' equal to 0. We get: $x = 0$ or $x = 1000$. So 1000 is the only critical point.

(c) Find all local and global extrema of $f(x)$.

ANSWER:
Use the first or the second derivative test for this. (The first derivative test is better in this case.)

(d) Use the above information to sketch a graph of $f(x)$. Be careful, your graphing calculator will very easily mislead you in this problem!