Math 110 | **BASIC CALCULUS I** | Fall 2007

Class 33

Friday November 30

Newton's Method

Finding Roots

We have seen that determining values c where $f'(c) = 0$ are important in finding local maxima and minima for f. Such a value is a *critical point* for f but a *root* for $g = f'$.

DEFINITION: root

The value r is a **root** of the function g if $g(r) = 0$.

There are many ways to find the roots of a function. One approach which sometimes works is to factor g into factors with known roots. In other cases you may have special knowledge of the function which you can use.

EXAMPLES

1. Find the roots of the function $g(x) = (x-3)^4(x+2)$.

2. Find the roots of the function $g(x) = x\ln(x)$.

3. Find the roots of the function $g(x) = \sin(2x)$.

Deriving Newton's Method from the Tangent Line

Consider a general function $h(t)$. We would like to get an approximation of a root, t^* of this function, and although we do not know the exact value of t^*, we can give a rough first approximation of $t = t_0$ near to $t = t^*$.

Consider the line tangent to $h(t)$ at $t = t_0$.

1. What is the slope of this line? (While we cannot write it down numerically, we do have notation that describes this slope.)

2. Name a point that this tangent line must pass through. (Give both coordinates.)

3. Using the slope and point of the tangent line, determine the equation of the tangent line (in the slope-intercept form $y = mt + b$).

$$y = \_\_\_\_\_\_\_\_\_\_ \; t + [\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_].$$

4. Now find the root $t = t_1$ of this tangent line, i.e., where the line crosses the t-axis. Simplify the expression for t_1 as much as possible. (Notice that your answer t_1 is dependent upon t_0, i.e. is a function of t_0.)

5. Suppose the process was repeated so that $t = t_{n+1}$ is the place where the tangent line to $h(t)$ at $t = t_n$ crosses the t-axis. Write down an expression for t_{n+1} in terms of t_n.

Deriving Newton's Method from the Microscope Approximation

Suppose we have obtained our nth approximation x_n for the root r of g, and we want to find a better approximation x_{n+1}. Ideally, we would like to know

$$\Delta x = r - x_n.$$

If we knew this exactly, then we could find the root r as $r = x_n + \Delta x$.

Since we don't know Δx exactly, we will appeal to the Microscope Approximation based at our current guess x_n:

$$\Delta y \approx g'(x_n)\Delta x \quad \Rightarrow \quad \Delta x \approx \frac{\Delta y}{g'(x_n)}.$$

This approximation is valid provided $g'(x_n) \neq 0$. But this will be true, provided x_n is sufficiently close to r, because we know that $g'(r) \neq 0$ and that g' is continuous on an open interval $a < r < b$ containing r.

Although we don't know Δx exactly, we do know Δy exactly! This is because we know $g(x_n)$ and we also know that $g(r) = 0$, so

$$\Delta y = \underline{\hspace{6cm}}.$$

Therefore,

$$\Delta x \approx \underline{\hspace{6cm}}$$

and we choose our next approximation x_{n+1} as

$$x_{n+1} = \underline{\hspace{6cm}}$$

Together with an initial guess x_0, this recursive process defines Newton's Method. Under the conditions listed above, we are guaranteed that

$$\lim_{n\to\infty} x_n = r.$$

i) an initial guess x_0 *sufficiently close* to the root r,

ii) $g'(r) \neq 0$ on an interval $a < x < b$ containing x_0 and r,

ii) g, g', and g'' continuous on an interval $a < x < b$ containing x_0 and r.

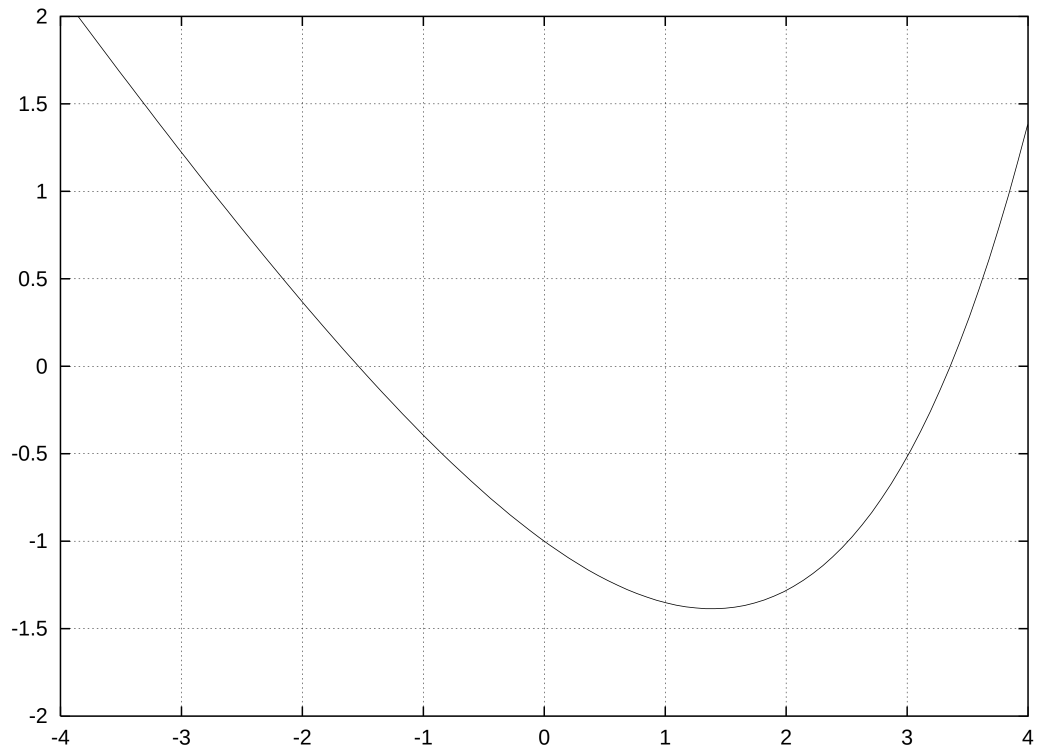
Provided these conditions are satisfied, Newton's Method is *guaranteed* to converge to the root r.

The big question, however, is "HOW CLOSE MUST x_0 BE TO r?" While certain formulas involving the second derivative can be given to address this question, in practice you simply run Newton's Method for a number of iterations to determine whether it is converging to the root you want. If you find it is not doing so, pick another value for x_0.

Visualizing Newton's Method

We derived Newton's Method by considering the equation of the tangent line to a function $h(t)$ at the point $t = t_0$ and then considering the root t_1 of this tangent line to be the approximation to the root t^* of the function $h(t)$.

Consider the graph of $h(t) = e^{t/2} - t - 2$ shown below on the interval $-4 \leq t \leq 4$.



1. Draw the tangent line to $h(t)$ at $t_0 = 1$. Extend the tangent line to the t-axis. Label this point t_1.

2. Draw the tangent line at $t = t_1$. Find its root and label this point t_2. Repeat as often as you can.

3. What do you notice about the sequence of points t_0, t_1, t_2, ...?

4. Visually, how does the limit of your sequence depend on the initial value t_0?

5. Are there any initial values which will cause the sequence to not converge?

4. Let $g(x) = x^2 - 17$. Confirm that g, g', and g'' are continuous (everywhere), and that $g'(x) \neq 0$ on an interval containing the root $r = \sqrt{17}$ and the initial guess $x_0 = 1$. Use three iterations of Newton's Method to approximate the root $r = \sqrt{17}$. For greatest accuracy, record your results as fractions or use the memory registers on you calculator to store intermediate results. Compare this with the value for $\sqrt{17}$ given by your calculator.

n	x_n	$\Delta x \approx -g(x_n)/g'(x_n)$	$x_{n+1} = \frac{1}{2}(x_n + 17/x_n)$
0	1		1
1			
2			
3			
4			
5			
6			