Basic Calculus I

Class 26

Friday November 9

Analyzing Graphical Behavior Of Functions

How do we connect what the graph of a curve looks like to the value of its derivative function at that point?

GroupWork

We are already aware that the derivative gives us qualitative information about the graphical behavior of the function. Fill in the blanks below:

function behavior	**derivative value**	**illustration**
increasing		
	negative	
level (flat)		
	undefined	
steep (rising or falling)		
	small (positive or negative)	
straight (horizontal)		
straight (slanted)		

The information from the table can be summarized in the following theorem

THEOREM

Let f be a funcion which is both continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

(a) $f'(x) > 0$ for every value of x in $(a, b) \Leftrightarrow f$ is **increasing** on $[a, b]$.

(b) $f'(x) < 0$ for every value of x in $(a, b) \Leftrightarrow f$ is **decreasing** on $[a, b]$.

(c) $f'(x) = 0$ for every value of x in $(a, b) \Leftrightarrow f$ is **constant** on $[a, b]$.

DEFINITION: Concavity

The second derivative $f''(x)$ gives us information about the **concavity** of the graph of $f(x)$. We say a graph of $f(x)$ is **concave up** on an interval if the graph stays above the tangent line at any point in the interval, and it is **concave down** if the graph stays below the tangent line at any point in the interval.
Let's draw pictures of concave up and concave down graphs.

THEOREM

Suppose a function is twice differentiable on a given open interval.

(a) $f'' > 0$ on an interval $\Leftrightarrow$ the graph of $f(x)$ is *concave up* on that interval.

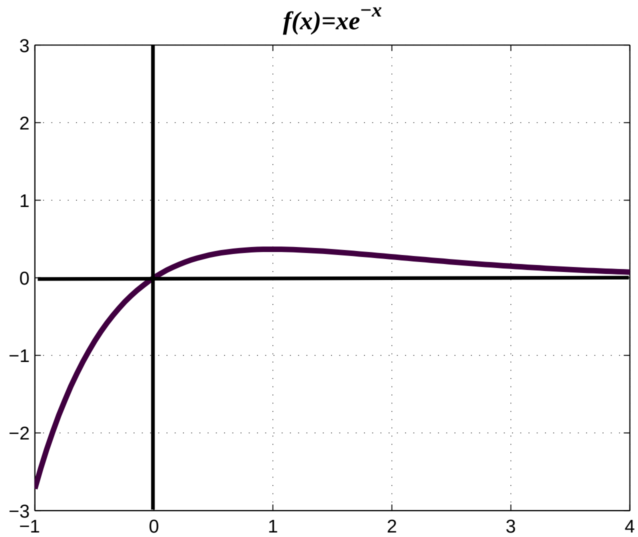
(b) $f'' < 0$ on an interval $\Leftrightarrow$ the graph of $f(x)$ is *concave down* on that interval

NOTE

Since the second derivative $f''(x)$ is the rate of change of the first derivative $f'(x)$ with respect to x this means that when $f''(x)$ is POSITIVE, $f'(x)$ must be ____________________ and when $f''(x)$ is NEGATIVE, $f'(x)$ must be ____________________.

EXAMPLE

Consider the graph of $f(x) = xe^{-x}$. Determine on what intervals the graph of the function will be concave up and concave down as well as what intervals the function will be increasing and decreasing.



DEFINITION: Inflection Point

If f is continuous on an interval and the concavity of the graph changes at a point $(c, f(c)$ we say that f has an **inflection point** or **point of infection** at c. Clearly, this would happen when $f''(x)$ has different signs when $x < c$ and whe $x > c$. The graphical interpretation of this change would be that f' switches from increasing to decreasing or vice versa.

Exercise

Find all the inflection points of $f(x) = x^4$.

EXAMPLE

Consider the graph of $f(x) = xe^{-x}$ (shown above). Does it have any inflection points?.

DEFINITION: Critical Points and Stationary Points

A point $(c, f(c))$ is called a **critical point** of a function $f(x)$ if $f^{'}(c) = 0$ or $f^{'}(c)$ does not exist. At these points the graph of $f(x)$ will have either have a horizontal tangent line or it will not be locally linear or it will have a vertical tangent line. A point $(c, f(c))$ at which $f'(c) = 0$ is called a **stationary point**. All stationary points are critical points. Not all critical points are stationary points.

EXAMPLE

AGAIN consider the graph of $f(x) = xe^{-x}$. Does it have any critical points?

CLICKER QUESTION

If a number very close to zero is divided by another number very close (but not equal) to zero, the result

(a) must be a number very close to zero.

(b) must be a number close to 1.

(c) could be any number.

(d) might not be a number at all.

CLICKER QUESTION

Consider the functions $f(x) = e^x$ and $g(x) = x^{1,000,000}$. As $x \to \infty$ which of the following is true?

(a) f grows faster than g.

(b) g grows faster than f.

(c) We cannot determine.

(d) They grow at the same rate like all exponentials.

CLICKER QUESTION

The limit $\lim_{x\to\infty} [xe^{1/x} - x]$

(a) Does not exist because $\infty - \infty$ is not defined.

(b) Converges to 1.

(c) Converges to 0.

(d) Is ∞ because $xe^{1/x}$ grows faster than x

CLICKER QUESTION

If a function is always positive, then what must be true about its derivative?

(a) The derivative is always positive.

(b) The derivative is never negative.

(c) The derivative is increasing.

(d) The derivative is increasing.

(e) You can't conclude anything about the derivative.

CLICKER QUESTION

The derivative, $f'(x)$, of a function $f(x)$ is negative everywhere. We also know that $f(0) = 0$. What must be true about $f(-1)$?

(a) $f(-1)$ is negative

(b) $f(-1)$ is positive

(c) $f(-1)$ is zero

(d) Not enough information to conclude anything about $f(-1)$