Math 110 | **Basic Calculus I** | Fall 2007

Class 24

Wednesday October 31

Derivatives of Inverse Functions

EXAMPLE

Given two functions $f(x)$ and $g(x)$ which have the relationship that $f(g(x)) = x$ and $g(f(x)) = x$ use the Chain Rule to show that $f'(g(x)) = \dfrac{1}{g'(x)}$ and $g'(f(x)) = \dfrac{1}{f'(x)}$.

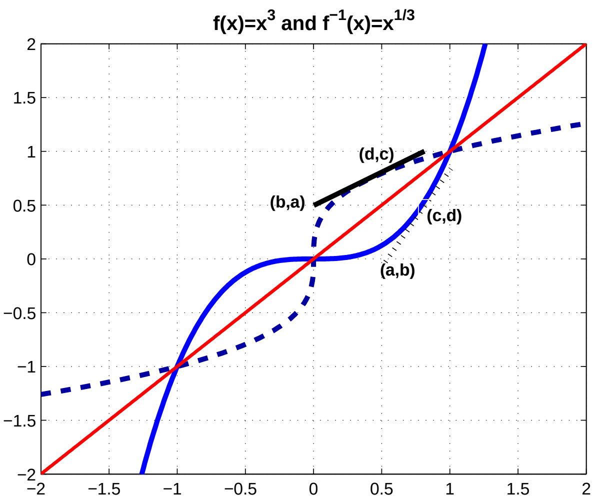
What is a word that we could use to describe $f(x)$ and $g(x)$?

Derivative of the Inverse Function

If $g = f^{-1}$, then $x = f(g(x))$. In the usual notation for inverse functions,

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}, \quad \text{provided } f'(f^{-1}(x)) \neq 0.$$

Graphical Approach



The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.
The line tangent to the graph of f^{-1} at (d, c) is the reflection across the line $y = x$ of the line tangent to the graph of f at (c, d).

If $(d, c) \neq (b, a)$ is on the line tangent to the graph of f^{-1} at (d, c), then $(c, d) \neq (a, b)$ is on the line tangent to the graph of f at (c, d).

The derivative of f^{-1} at b can then be computed as the slope of the line tangent to the graph of f^{-1} at (b, a):

$$(f^{-1})'(d) = \frac{c-a}{d-b} = 1\Big/\left(\frac{d-b}{c-a}\right) = 1/f'(c),$$

the reciprocal of the slope of the line tangent to the graph of f at (c, d).

Another View Another way of thinking about this result is if $y = f^{-1}(x)$ then by definition of inverses $x = f(y)$.

$$\frac{dy}{dx} = \frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(y))} = \frac{1}{\frac{dx}{dy}}$$

EXAMPLE

Let's demonstrate the above result with $f(x) = x^3 + 1$.

THEOREM: monotonically increasing/decreasing functions are invertible

A function $f(x)$ that has a derivative $f'(x)$ which is either always positive or always negative for every value in the domain of $f(x)$ is said to be monotonically increasing (when $f' > 0$) or monotonically decreasing (when $f' < 0$). Monotonically increasing functions are one-to-one (invertible) functions and their inverse functions f^{-1} are differentiable at every value in the range of $f(x)$.

GroupWork

Consider $f(x) = x^5 + x + 1$.

1. Show that f is one-to-one on the interval $(-\infty, \infty)$.

2. Show that f^{-1} is differentiable on the interval $(-\infty, \infty)$.

3. Find a formula for f^{-1} using implicit differentiation

4. Find a formula f^{-1} using another method