Class 22

Friday October 26

Differentiation of Logarithmic Functions

THEOREM

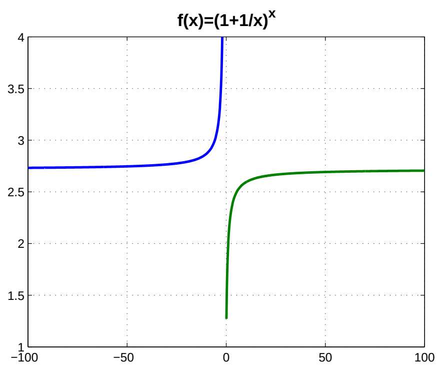
Taking the limit of a function $f(x)$ as its input increases without bound **positively** $(x \to +\infty)$ is identical to taking the limit of $f(1/x)$ as x approaches zero from the right $(x \to 0^+)$. Similarly, Taking the limit of a function $f(x)$ as its input increases without bound **negatively** $(x \to +\infty)$ is identical to taking the limit of $f(1/x)$ as x approaches zero from the left $(x \to 0^-)$. Mathematically, we can write this as

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(1/x) \text{ and } \lim_{x\to -\infty} f(x) = \lim_{x\to 0^-} f(1/x)$$

This means that when $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = L$ we also know $\lim_{x\to 0} f(1/x) = L$

This is a very useful result because it allows us to swap an infinite limit (which might be hard to compute) for a limit to a finite number (zero!) which could be easier to compute.

Consider the function $f(x) = (1+\frac{1}{x})^x$. It turns out that the graph of $f(x)$ possesses the horizontal asymptote $y = e \approx 2.7182818\ldots$ as $x \to +\infty$ **and** $x \to -\infty$



EXAMPLE

We can use this information to show that $\lim_{t\to 0}(1+t)^{1/t} = e$.

We can use this limit result plus the limit definition of the derivative to evaluate $\frac{d}{dx}[\ln(x)]$.

THEOREM

When $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}, x > 0$ and when $f(x) = \log_b(x)$, $f'(x) = \frac{1}{x\ln(b)}, x > 0$

Exercise

Evaluate $\frac{d}{dx}\ln(x^2+1)$ and $\frac{d}{dx}x^2\ln(5x)$

EXAMPLE

Let's prove that $\frac{d}{dx}\ln|x| = \frac{1}{x}, x \neq 0$ by considering two cases, $x > 0$ and $x < 0$.

Logarithmic Differentiation

The technique of logarithmic differentiation is used when an function $y = f(x)$ has a lot of products, powers and quotients so that $\ln y = \ln(f(x))$ is actually easier to differentiate (as long as we also then use the Chain Rule).

Suppose $y = x^r$ where r is **any** real number. Let's use logarithmic differentiation to prove $(x^r)' = rx^{r-1}$.

GROUPWORK

1. Evaluate $\frac{d}{dx}[x^{\pi}]$.

2. $f(x) = \frac{\sin(x)}{x^{\sqrt{2}}}$, find $f'(x)$.

3. Evaluate (a) $\lim_{h\to 0}\frac{\ln(\cos(h))}{h}$ and (b) $\lim_{w\to 1}\frac{\ln(w)}{w-1}$

4. *Anton, Bivens & Davis*, **Question 4.2.37**

Find $\frac{d}{dx}[\log_x e]$ and $\frac{d}{dx}[\log_x 2]$.