Math 110 — Fall 2007

Basic Calculus I

Class 12

Wednesday September 26

Continuity Continued

THEOREM: **Continuity of Trigonometric Functions**

We have previously said that $\cos(x)$ and $\sin(x)$ are continuous everywhere, i.e. on the domains $(-\infty, \infty)$. It turns out that all trigonometric functions are continuous on their natural domains, so that:

(a) $\lim_{x \to a} \sin(x) = \sin(a), -\infty < a < \infty$

(b) $\lim_{x \to a} \cos(x) = \cos(a), -\infty < a < \infty$

(c) $\lim_{x \to a} \tan(x) = \tan(a), -\pi/2 < a < \pi/2$

(d) $\lim_{x \to a} \cot(x) = \cot(a), -\pi/2 < a < 0 \cup 0 < a < \pi/2$

(e) $\lim_{x \to a} \sec(x) = \sec(a), a \neq (2n+1)\pi/2, \quad \forall n$

(f) $\lim_{x \to a} \csc(x) = \csc(a), a \neq (2n)\pi/2, \quad \forall n$

THEOREM: **Continuity of Inverse Functions**

If $f(x)$ is a one-to-one (i.e. invertible) function that is continuous on every point in its domain, then the inverse function $f^{-1}(x)$ is continuous in its domain. Recall, the domain of f^{-1} is the range of f and vice versa.

Example

Is $\sin^{-1}(x)$ continuous on its domain of $[-1, 1]$?

Exercise

Show that $\tan^{-1}(x)$ is everywhere continuous.

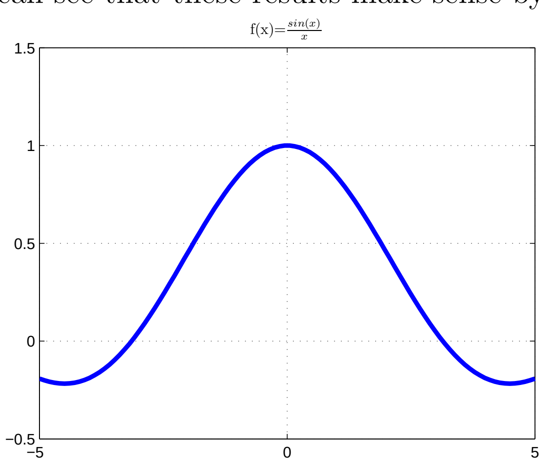
THEOREM

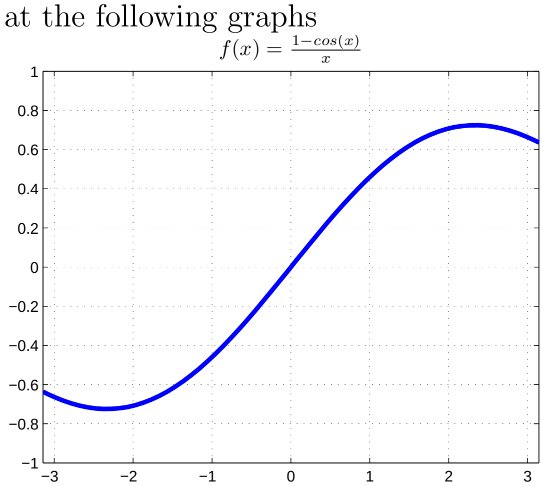
Here are some important unusual limits we will use later on in the class

(a) $$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

(b) $$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$$

We can see that these results make sense by looking at the following graphs





The texbook proves the values of these limits by applying an important thoerem called The Squeeze Theorem.

THEOREM: The Squeeze Theorem

Let $f(x), g(x)$ and $h(x)$ be functions satisfying the inequality

$$g(x) \leq f(x) \leq h(x)$$

for every x in an open interval containing the number a with the possible exception that the inequalities DO NOT have to hold at $x = a$. IF $g(x)$ and $h(x)$ have the same limit as x approaches a, i.e.

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

THEN $f(x)$ also has this same limit as x appproaches a,

$$\lim_{x \to a} f(x) = L$$

We can use the Squeeze Theorem to prove that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

once we first use a little trigonometry to show that for $0 < x < \frac{\pi}{2}$, $\frac{1}{2}\tan(x) \geq \frac{1}{2}x \geq \frac{1}{2}\sin(x)$

$\frac{1}{2}\tan(x) \geq \frac{1}{2}x \geq \frac{1}{2}\sin(x)$	See Diagram Above
$\frac{1}{\cos(x)} \geq \frac{x}{\sin(x)} \geq 1$	Divide Each Term by $\sin(x)$ which is not zero in the range $0 < x < \pi/2$
$\cos(x) \leq \frac{\sin(x)}{x} \leq 1$	Take The Reciprocal Of Each Term, Flips The Inequalities

Since $\lim_{x \to 0} \cos(x) = 1$ and $\lim_{x \to 0} 1 = 1$. We can use the Squeeze Theorem to say that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

Then we can use $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ and Properties of Limits to show that $\lim_{x \to 0} \frac{1 - \cos(x))}{x} = 0$ (See Page 158, of *Anton, Bivens & Davis*).