Math 110

Basic Calculus I

Class 11

Monday September 24

Introduction to Continuity

Fall 2007

An informal definition of continuity of a function is that if it's graph can be drawn without picking up your drawing implement, then it is continuous at all the iput values that are graphed.

DEFINITION: **Continuity at a point**

A function $f(x)$ is said to be **continuous at a point** $x = a$ if ALL of the following statements are true:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

If a function is not continuous at a point we say that it has a **discontinuity at** $x = a$.

EXAMPLE

Let's think about the different ways that this defnition can fail for a function:

1. $f(x)$ may be undefined at the point $x = a$
2. The value of $f(a)$ and the value of the limit of $f(x)$ as x approaches a may be different
3. The limit of $f(x)$ as x approaches a may not exist because the one sided limits are different
4. The limit of $f(x)$ as x approaches a may not exist for other reasons than those given in (3)

GroupWork

Let's provide algebraic examples of functions which correspond to each of the "Discontinuity Scenarios" described above and draw a picture of each of the scenarios which depict the way a function can be discontinous at a apoint.

DEFINITION: **Continuity on an open interval**

If a function $f(x)$ is continuous at every point in an open interval (a, b) we say that $f(x)$ **is continuous on the interval** (a, b). If $f(x)$ is continuous at every point in the interval $(-\infty, \infty)$ we say that $f(x)$ **is continuous everywhere.**

Examples of Everywhere Continuous Functions

One very nice feature of polynomials was that $\lim_{x \to a} p(x) = p(a)$. This means that polynomial functions are continuous everywhere.

Q: Can you think of at least **three** other families of functions which are continuous everywhere?

A: ____

Rational Functions

A rational function $\dfrac{p(x)}{q(x)}$ is continuous at every point at which the denominator function $q(x)$ is NOT equal to zero, and possesses discontinuities at al the points where $q(x) = 0$.

Exercise

For what values of x is the function $f(x) = \dfrac{x^2 - 9}{x^2 - 5x + 6}$ continuous? Express your answer using interval notation.

Continuity of Composite Functions

THEOREM

If a function $g(x)$is continuous at a point $x = a$ such that $\lim_{x \to a} g(x) = g(a)$ and a function $f(x)$ is continuous at the point $x = g(a)$ then $\lim_{x \to a} f(g(x)) = f(g(a))$.

This is a very important theorem because it means that when taking the limit of a composite function one can "bring in" the limit process from the outer function to the inner function, i.e.

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(g(a))$$

This Theorem is of course most useful for functions which are continuous everywhere, which results in the following correspnding theorem: If f and g are functions that are continuous everywhere, then $f \circ g$ is continuous everywhere.

GroupWork

Which are the following functions are continuous everywhere?

e^{x^2}, $|\sin(x)|$, $\dfrac{1}{\cos(x)}$, $(x^5 + 6)^8$, $(x^2 + 2x + 1)^{-7}$, $\sqrt{x^5 + 7}$

CLICKER QUESTION

TRUE or FALSE: At some point in time, you were once exactly 3 feet tall.

(a) TRUE

(b) FALSE

CLICKER QUESTION

Suppose that during half-time at a basketball game, the score of the home team is 36 points. TRUE or FALSE: At some point in time during the first half, the home team must have had exactly 25 points.

(a) TRUE

(b) FALSE

CLICKER QUESTION

TRUE or FALSE: At some point in time, your weight in pounds was exactly equal to your height in inches.

(a) TRUE

(b) FALSE

CLICKER QUESTION

You know that following statement is TRUE: **IF $f(x)$ is a polynomial, THEN $f(x)$ is continuous everywhere.** Which of the following is also true?

(a) IF $f(x)$ is NOT continuous everywhere, THEN it is not a polynomial

(b) IF $f(x)$ is continuous everywhere, THEN it is a polynomial

(c) IF $f(x)$ is NOT a polynomial, THEN it is NOT continuous

(d) None of the above

THEOREM

Suppose that the functions $f(x)$ and $g(x)$ are continuous at $x = a$, then

(a) The sum of two functions continuous at a point is continuous at that point:
$f + g$ is continuous at $x = a$.

(b) The difference of two functions continuous at a point is continuous at that point:
$f - g$ is continuous at $x = a$.

(c) The product of two functions continuous at a point is continuous at that point:
$f \cdot g$ is continuous at $x = a$.

(d) The quotient of two functions continuous at a point is continuous at that point:
f/g is continuous at $x = a$ if $g(a) \neq 0$ and f/g has a discontuity at $x = a$ if $g(a) = 0$.

DEFINITION: Continuity on a closed interval

In order to define continuity on a closed interval we have to define the notion of "continuity from the left at a point" and "continuity from the right at a point." A function f is continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$. A function f is continuous from the left at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$. Then we can say that a function $f(x)$ is continuous on a closed iterval $[a, b]$ if A function $f(x)$ is said to be **continuous at a point** $x = a$ if ALL of the following statements are true:

1. $f(x)$ is continuous on the open interval (a, b)
2. $f(x)$ is continuous from the right at $x = a$
3. $f(x)$ is continuous from the left at $x = b$

THEOREM: The Intermediate Value Theorem

If $f(x)$ is continuous on a closed interval $[a, b]$ and k is *any* number between $f(a)$ and $f(b)$ inclusive, then there is AT LEAST one number x in the interval $[a, b]$ such that $f(x) = k$.
The main way that the Intermediate Value Theorem is used is in proving the following theorem, which is also extremely important.

THEOREM

If $f(x)$ is continuous on a closed interval $[a, b]$ and $f(a)$ and $f(b)$ are non-zero and have opposite signs, then there is AT LEAST one solution of the equation $f(x) = 0$ in the interval (a, b).
Let's draw a picture of this situation. This is a powerful result which allows us to develop algorithms to obtain the root of a function.