Class 5

Monday September 10

Inverse Functions

Inverses and Identities as Theoretical Objects

Many operations on sets of numbers or functions possess both an *identity element* and an *inverse element* in the set. Important examples of these operations include addition, multiplication, and composition of functions.

Addition

There is exactly one real number a with the property that

$$a + x = x + a = x, \quad \text{for all } x \in \mathbb{R}.$$

This number, the *additive identity*, is $a = 0$.

If b is a real number, there is exactly one real number c such that

$$b + c = c + b = 0.$$

This number, the *additive inverse* of b, is $c = -b$.

We can extend these ideas from numbers to functions. There is exactly one function $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) + f(x) = f(x) + h(x) = f(x), \quad \text{for all } f : \mathbb{R} \to \mathbb{R}.$$

This function, the *additive identity* for functions, has the formula $h(x) = 0$, for all $x \in \mathbb{R}$.

The *additive inverse* of the function f is the function g such that

$$f(x) + g(x) = g(x) + f(x) = h(x) = 0.$$

In fact, $g(x) = -f(x)$. There is a visual interpretation of the additive inverse of a function. The graph of $-f(x)$ is obtained by reflecting the graph of $f(x)$ across the x-axis.

Multiplication

There is exactly one real number a with the property that

$$a \cdot x = x \cdot a = x, \quad \text{for all } x \in \mathbb{R}.$$

This number, the *multiplicative identity*, is $a = 1$.

If $b \neq 0$ is a real number , there is exactly one real number c such that

$$b \cdot c = c \cdot b = 1.$$

This number, the *multiplicative inverse* of b, is $c = b^{-1} = 1/b$, the *reciprocal* of b.

NOTE

the existence of the multiplicative inverse is conditional; it only happens if $b \neq 0$.
We can extend these ideas from numbers to functions. There is exactly one function $k : \mathbb{R} \to \mathbb{R}$ such that

$$k(x) \cdot f(x) = f(x) \cdot k(x) = f(x), \quad \text{for all } f : \mathbb{R} \to \mathbb{R}$$

This function, the *multiplicative identity* for functions, has the formula $k(x) = 1$, for all $x \in \mathbb{R}$.

The *multiplicative inverse* of the function f is the function g such that

$$f(x) \cdot g(x) = g(x) \cdot f(x) = k(x) = 1.$$

In fact, $g(x) = [f(x)]^{-1} = 1/f(x)$, which exists for all x in the domain of f for which $f(x) \neq 0$.

Composition of Functions

This operation has no counterpart for real numbers. Recall that $(f \circ g)(x) = f(g(x))$. The *identity function* (under composition) is the function $\mathcal{I} : \mathbb{R} \to \mathbb{R}$ such that

$$(f \circ \mathcal{I})(x) = (\mathcal{I} \circ f)(x) = f(x), \quad \text{for all } f : \mathbb{R} \to \mathbb{R}.$$

The formula for $\mathcal{I}$ is $\mathcal{I}(x) = x$, for all $x \in \mathbb{R}$.

The function g is the *inverse* of f (under composition) if

$$(f \circ g)(x) = (g \circ f)(x) = \mathcal{I}(x) = x,$$

That is, if g is the inverse of f under composition, then $f(g(x)) = g(f(x)) = x$ for all x in the domain of f. The inverse of f under composition is generally denoted by $f^{-1}(x)$ and is generally known as "THE inverse" of $f(x)$. The domain of the inverse of $f(x)$ is the same exact set as the range of $f(x)$, and the range of the inverse of $f(x)$ is the domain of $f(x)$.

NOTE **In general, the multiplicative inverse of f is not the inverse of f:**

$$[f(x)]^{-1} \neq f^{-1}(x)$$

Conceptualizing The Inverse Function Practically

Let's think of the function $f(x) = 3x + 5$ as some kind of "machine" with gears in it:
You input a number on the left, the gears turn and turn, and to the right the machine gives its output!
If you input 4 on the left, the machine will output ________ on the right.
We write this as: $f(4) =$
Now imagine we hit the "reverse button" on the function machine: all the gears will turn in the opposite direction!
If you input 11 on the RIGHT, the machine will output ______ on the LEFT.
We write this as: $f^{-1}(11) =$

NOTE 2^{-1} means 1/2, which equals 0.5. Similarly, x^{-1} means $1/x$. **Sadly $f^{-1}(11)$ does NOT mean $\frac{1}{f(11)}$!!!**

So what does $f^{-1}(11)$ mean?
When we ask $f^{-1}(11) =?$, we're just asking $f(?) = 11$.

EXAMPLE

Now let's think of the function $f(x) = 3x + 5$ as a RULE, or a set of instructions:
The function's rule says:
Whatever INPUT you give me, first I will __________________ it by __________, and then I will _________________________ to the answer and OUTPUT ____________
Now describe a RULE for $f^{-1}(x)$, i.e., a RULE which takes the output of f and transforms it back to the original input which became that output.
HINT: Do each operation in reverse!
Whatever INPUT you give me, first I will __________________ it by __________, and then I will _________________________ to the answer and OUTPUT ____________
Mathematically, we write the formula for $f^{-1}(x) =$ ______________________.
Let's confirm that our formula for $f^{-1}(x)$ is indeed the inverse of $f(x)$ by composing the two functions.
$(f \circ f^{-1})(x) =$ ____________________________________
$(f^{-1} \circ f)(x) =$ ____________________________________

Exercise

Fill in the blanks.
$g(x) = (x+3)^2$. Describe its RULE below:
Whatever number you give me, first I will _____________, then I will ____________________ the answer, and then I will ________ the answer.
Now describe a RULE for $g^{-1}(x)$.

Convert your RULE into a **FORMULA** for $g^{-1}(x)$.

Existence of Inverse

Q: Does every function have an inverse? **A:** No!
Q: How do we determine whether a function has an inverse? **A:** When each OUTPUT value has a SINGLE input value.
Since by definition a function assigns each input a single output, an *invertible* function is said to be a **one-to-one** function.
Graphically, one can determine whether a function is invertible if it passes the Horizontal Line Test. In other words, if the graph of a function is intersected no more than once by any horizontal line, then that function is **invertible**.

GroupWork

In the space below, draw the graph of a function which is invertible and the graph of a function which is not invertible.

THEOREM

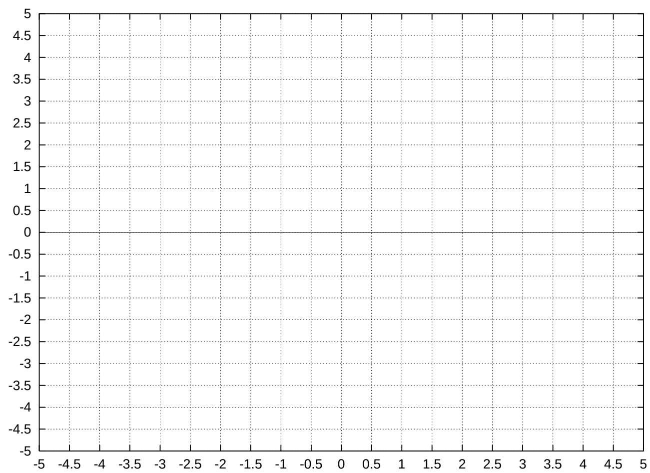
The inverse of a given function $f(x)$ exists if and only if $f(x)$ is one-to-one. The graph of a function will intersect any horizontal line once if and only if it is an invertible function (This is known as the Horizontal Line Test).

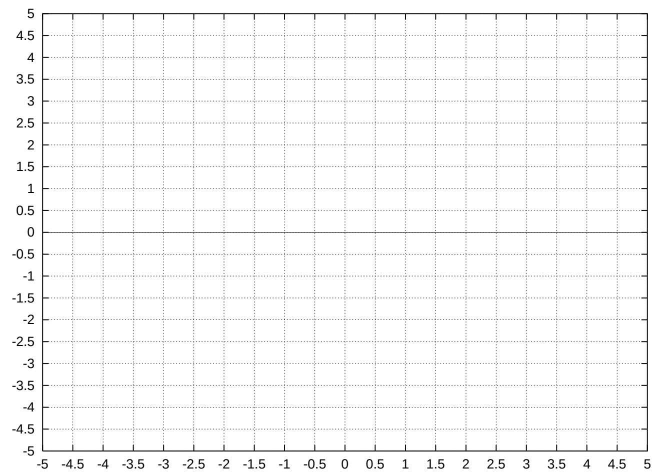
Graphical Interpretation of Inverse Functions

1. Suppose $f(a) = b$. (This means that the point (a, b) is on the graph of f.)
Show that $f^{-1}(b) = a$. (This means that the point (b, a) is on the graph of f^{-1}.) How are the points (a,b) and (b,a) related to each other?

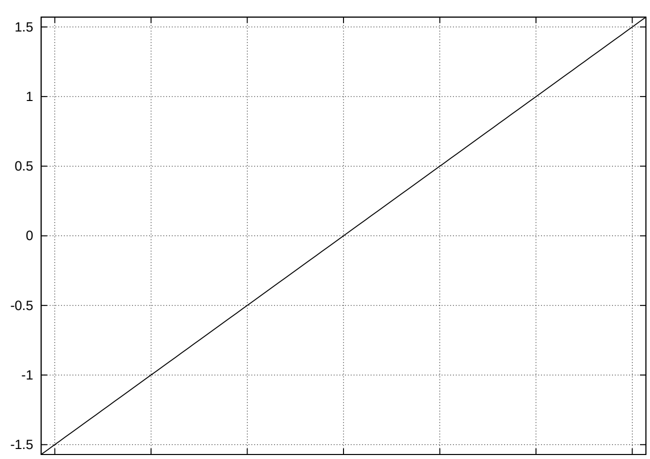
2. Use this result to explain why the graph of f^{-1} is the *reflection about the line* $y = x$ of the graph of f.

Example: On one figure, sketch graphs of $f(x) = 3x+5$ *and its inverse* $f^{-1}(x)$ *on the same axes. On the other figure, sketch graphs of* $g(x) = (x+3)^2$ *and its inverse* $g^{-1}(x)$ *on the same axes. What do you notice?*





Example: Sketch graphs of $g(x) = \sin(x),\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $\quad g^{-1}(x) = \arcsin(x) := \sin^{-1}(x),\ -1 \le x \le 1$.



NOTE *the restriction of the domain of each function. Why is this necessary?*