Lab 6: Limits, Continuity, and Differentiability

Objectives:

1. To become familiar with the program **Derive** as a tool (especially to evaluate limits).
2. To become more familiar with finding the derivative of a function at a point by taking limits of difference quotients.
3. To better understand the relationship between differentiability and continuity.

§1 Introduction: Using *Derive* To Explore Graphs

Derive has rather nice features for exploring graphs of functions. This is what we will use it for today. To begin, click on the *Derive* icon under *Mathematics*. A screen will appear with a list of menu options and buttons at the top. This particular screen is called the "Algebra" window in *Derive* because this window will be used to author and modify algebraic and other expressions.

Type

$$\sin(1/x)$$

in the authoring window, then < *Enter* > it. The authoring window will disappear and your expression will appear in the algebra window.

To plot the function whose rule is given by an expression, make sure the expression is *highlighted* in the algebra window, then click on the second button from the right in the toolbar. Do this now.

The screen is now replaced with a "graphics" window. This window will have a pair of axes marked with tickmarks and its own menu at the top. Now select and enter **Plot** from this menu. The graph of this function should appear.

Derive has several features which allow you to explore graphs. First, notice the cross-hairs. They can be controlled with either the mouse or the "arrow" keys. At the bottom of the screen you will see the x- and y-coordinates changing as you move the cross-hairs around.

Now examine the bottom of the screen more closely. The spacing between the tickmarks on the x and y axes will appear as **Scale** in the format *x-scale*: *y-scale*. What are these values now?

There are several features of the graphics window menu which we will also be using. Select **Set**, then **Center**. Type 0 for the *Horizontal* coordinate and 1 for the *Vertical* coordinate, then < *Enter* >. Describe what happens. (Also note the *Center* box at the bottom of the screen.)

The other feature we will be using is **Zoom**. Various sorts of zooming are possible. These are performed by the buttons at the right side of the menu bar with little arrows on them. Find and select the button which *zooms in on both axes.* Describe what happens. Pay particular attention to the values for the x and y scales.

You now know the basics of working with *Derive*. During the rest of the lab, we will be using the following sequence of operations to focus on certain points of the graph of a function.

Move the cross-hairs to the point of interest.

Center *the window on that point.*

Zoom *in on the center of the window.*

Try zooming in and out on various points just to get the hang of this sequence of operations.

§2 Continuity

1. Consider the function $p(x) = \sin(x)/x$. Use *Derive* and/or your calculator to estimate $\lim_{x \to 0} p(x)$. Look at $p(x)$ for the following sequences of x values approaching $x = 0$ from the left and from the right.

x	$p(x)$	x	$p(x)$
-1		1	
-0.1		0.1	
-0.01		0.01	
-0.001		0.001	
-0.0001		0.0001	

Do you think you would get the same results for other sequences of x values approaching 0 from below or from above? *Zooming in* on the graph with *Derive* may help you decide. Based on your conclusion, determine the following limit or explain why it does not exist.

$$\lim_{x \to 0} p(x) =$$

Informally, a function is *continuous* at a point if its graph is unbroken there. Another way to think of this is: a function is continuous if you can draw its graph without lifting the pencil off the paper. This idea can also be expressed in terms of limits.

Definition: A function $g(x)$ is *continuous* at a if $\lim_{x \to a} g(x) = g(a)$.

2. Based on this definition, is $p(x) = \sin(x)/x$ continuous at $x = 0$? *Hint: What is the value of $p(x)$ at $x = 0$?*

3. Complete the following definition so that the function $q(x)$ *is* continuous at $x = 0$:

$$q(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ & \text{if } x = 0 \end{cases}$$

4. Is $x = 0$ a removable discontinuity of $p(x)$, $q(x)$, both or neither?

§3 Differing Difference Quotients

We have previously defined the derivative of $f(x)$ at a point a using the *forward difference formula*

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

If this limit exists then we say that $f'(a)$ exists and is equal to these limits.
IF you know that $f'(a)$ exists, you can also calculuate it using the more accurate *centered-difference formula*

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}.$$

Use *D*erive to graph $y = f(x) = x^3 - 1$ on with an x range of $[0, 2]$ and a y-range of $-2, 2]$.

We will use the table below to investigate what happens when you find the derivative of $f(x) = x^3 - 1$ at $x = 1$ using the different difference quotients. You can use *D*erive to help you complete the table by defining a function. If you *Author* `RIGHT(h) := (f(1+h) - f(1))/h` (don't forget the colon or ANY of the parentheses!) then you now have a function `RIGHT(h)` which you can use to complete the first table below. Similarly define a `LEFT(h)` function to help you complete the second table.

h	$1+h$	$f(1+h)$	$\frac{f(1+h)-f(1)}{h}$
0.1			
0.01			
0.001			
0.0001			

h	$1-h$	$f(1-h)$	$\frac{f(1)-f(1-h)}{h}$
0.1			
0.01			
0.001			
0.0001			

Use your answers from above to help you complete the following table.

h	$\frac{f(1+h)-f(1-h)}{2h}$
0.1	
0.01	
0.001	
0.0001	

Do you expect the results of the three limits to agree? Do they? What is the value of $f'(1)$ for $f(x) = x^3 - 1$?

Write down the equation of the tangent line to $f(x) = x^3 - 1$ at $x = 1$.

Use *D*erive to show the function $f(x)$ and its tangent line on the same graph. Zoom in on the point where the tangent line and curve intersect. Can you zoom in enough so that the difference between the two graphs is negligible? What does this tell you about the relationship between *local linearity* and differentiability?

§4 The Derivative of the Absolute Value Function

The absolute value function is denoted by "abs(x)" in *Derive*. Use *Derive* and the methods you have just learned to first obtain a graph of $y = f(x) = |x|$, and then estimate the slope of the graph at $x = 0$.

h	$\frac{f(0+h)-f(0)}{h}$	$\frac{f(0-h)-f(0)}{-h}$	$\frac{f(0+h)-f(0-h)}{2h}$
0.1			
0.01			
0.001			

Do you think it is possible to define the slope of the graph of the absolute value function at $x = 0$? Why or why not? How does this result affect your understanding of the relationship between differentiability and continuity?

§5 Using Derive to Evaluate Limits

Consider the difference quotients when $f(x) = |x|$. Simplify the expressions algebraically and then write down the *Derive* formula for the simplified form of the difference quotients.

$$\frac{f(0+h)-f(0)}{h} =$$

$$\frac{f(0+h)-f(0-h)}{2h} =$$

If you *Author* each expression in *D*erive and then select the `Lim` button you will get a window which allows you to take limits with respect to different variables and from either the "left," "right" or "both." Be careful you understand what variable is used in each of the examples above. Take the limit from the left and then the right and check to see how they correspond with the limit from "both" for each difference quotient above. Does this confirm the results you obtained using the "tabular method" of computing limits?

§6 Using Excel to Evaluate Limits

Consider the Spreadsheet in the directory `S:\Math Courses\Math110` named `Wk06F2007Lab.xls`. There are two "sheets" named **Limits** and **Difference Quotients**. Can you figure out which cells you need to change on the difference quotient spreadsheet in order to let it produce forward, backward and centered difference quotient estimates of the derivative of $f(x) = x^3 - 1$ at $x = 1$?

Preparing Your Lab Report

Your report should consist of a cover page with the names and signatures of your lab group members as well as what lab section this report is being submitted for. You should use the **Grade Allocation for Lab Team Writing Assignments** sheet to provide this information. In addition, you should submit some information on the group dynamics, providing the details on how you were able to work together to produce the final report.
Each person (in a group of three) should complete a first draft of one of the three parts, and the group should meet (*at least* once) to read and discuss these drafts before submitting the final lab report. The final report is due at the beginning of lab in TWO WEEKS: **Thursday October 18**. Your lab report should be word-processed and grammatically coherent!

Your lab report should take care to answer the following *specific* questions. Be sure to provide the context for each question in your answer.

Part 1
Consider the graph of $\sin(1/x)$. Is $\sin(1/x)$ continuous at $x = 0$? Is it differentiable at $x = 0$? Give evidence to support your answers to these questions. How is the behavior of $\sin(1/x)$ at $x = 0$ different from the behavior of the function $\sin(x)/x$ at $x = 0$? Discuss.

Part 2
Consider the expressions for `RIGHT(h)`$= \dfrac{f(1+h) - f(1)}{h}$ and `LEFT(h)`$=\dfrac{f(1+(-h)) - f(1)}{-h}$. Algebraically, show that `LEFT(h) = RIGHT(-h)`. Also, show that the average of `LEFT(h)` and `RIGHT(h)` is exactly the centered-difference formula, `CENTER(h)` $= \dfrac{f(1+h) - f(1-h)}{2h}$.
Why is it that the derivative at a point is not defined using the centered-difference formula? (HINT: What evidence do you have from this lab that using the centered-difference formula to find the derivative does not always give you the correct answer for the derivative of a function at a point? Think about the behavior of the absolute value function at $x = 0$.)

Part 3
Consider the rate of convergence of the centered-difference formula to the derivative. Does your tabular evidence support the claim that taking the limit using this formula converges to the derivative faster than the difference quotient used to define the derivative does? Do you believe this statement is true? Try graphically depicting what each method looks like for the function $f(x) = x^3 - 1$ at $x = 1$ to support your claim.