(b) For a fixed value of \( n \) we have, for \( k = 1, 2, \ldots, n \), \( y_k = \frac{y_{k-1} + y_{k-1}}{n} \). In particular \( y_n = \left( \frac{n+1}{n} \right)^n y_{n-1} = \left( \frac{n+1}{n} \right)^{n-1} y_{n-2} = \ldots = y_0 = \left( \frac{n+1}{n} \right)^0 \). Consequently, 

\[
\lim_{n \to +\infty} y_n = \lim_{n \to +\infty} \left( \frac{n+1}{n} \right)^n = e, \text{ which is the (correct) value } y = e^{\left| x = 1 \right|}.
\]

**EXERCISE SET 9.3**

1. (a) \( \frac{dy}{dt} = ky^2, \ y(0) = y_0, k > 0 \)

2. (a) \( \frac{ds}{dt} = \frac{1}{2} \) \( s \)

3. (a) \( \frac{dy}{dt} = 0.02y, \ y_0 = 10,000 \)

4. (a) \( \frac{dv}{dt} = -2v^2 \)

5. (a) \( \frac{dy}{dt} = 0.02y, \ y_0 = 10,000 \)
   
   (c) \( T = \frac{1}{0.02} \ln 2 \approx 34.657 \text{ h} \)

6. \( k = \frac{1}{T} \ln 2 = \frac{1}{20} \ln 2 \)
   
   (a) \( \frac{dy}{dt} = (\ln 2/20)y, \ y(0) = 1 \)
   
   (c) \( y(120) = 2^6 = 64 \)

7. (a) \( \frac{dy}{dt} = -ky, \ y(0) = 5.0 \times 10^7; \ 3.83 = T = \frac{1}{k} \ln 2, \text{ so } k = \frac{\ln 2}{3.83} \approx 0.1810 \)
   
   (b) \( y = 5.0 \times 10^7 e^{-0.181t} \)
   
   (c) \( y(30) = 5.0 \times 10^7 e^{-0.1810(30)} \approx 219,000 \)
   
   (d) \( y(t) = (0.1)y_0 = y_0 e^{-kt}, \ -kt = \ln 0.1, \ t = -\frac{\ln 0.1}{0.1810} = 12.72 \text{ days} \)

8. (a) \( k = \frac{1}{T} \ln 2 = \frac{1}{140} \ln 2 \approx 0.0050, \text{ so } \frac{dy}{dt} = -0.0050y, \ y_0 = 10. \)
   
   (b) \( y = 10e^{-0.0050t} \)
   
   (c) \( 10 \text{ weeks} = 70 \text{ days} \text{ so } y = 10e^{-0.35} \approx 7 \text{ mg} \)
   
   (d) \( 0.3y_0 = y_0 e^{-kt}, \ t = -\frac{\ln 0.3}{0.0050} \approx 240.8 \text{ days} \)
9. \(100e^{0.02t} = 10000, \quad e^{0.02t} = 100\), \(t = \frac{1}{0.02} \ln 100 \approx 230\) days

10. \(y = 10000e^{kt}\), but \(y = 12000\) when \(t = 5\) so \(10000e^{5k} = 12000\), \(k = \frac{1}{5} \ln 1.2\). \(y = 20000\) when

2. \(e^{kt}\), \(t = \frac{\ln 2}{k} = 5\ln \frac{2}{\ln 1.2} \approx 19\), in the year 2017.

11. \(y(t) = y_0e^{-kt} = 10.0e^{-kt}, \quad 3.5 = 10.0e^{-k(9)}, \quad k = -\frac{1}{5} \ln \frac{3.5}{10.0} \approx 0.2100, \quad T = \frac{1}{k} \ln 2 \approx 3.30\) days

12. \(y = y_0e^{-kt}, \quad 0.7y_0 = y_0e^{-sk}, \quad k = -\frac{1}{5} \ln 0.7 \approx 0.07\)

(a) \(T = \frac{\ln 2}{k} \approx 9.90\) yr

(b) \(y(t) \approx y_0e^{-0.07t}, \quad \frac{y}{y_0} \approx e^{-0.07t}\), so \(e^{-0.07t} \times 100\) percent will remain.

13. (a) \(k = \frac{\ln 2}{6} \approx 0.1155; y \approx 3e^{0.1155t}\)

(b) \(y(t) = 4e^{0.02t}\)

(c) \(y = y_0e^{kt}, \quad 1 = y_0e^k, \quad 200 = y_0e^{10k}\). Divide: \(200 = e^k, \quad k = \frac{1}{9} \ln 200 \approx 0.5887, \quad y \approx y_0e^{0.5887t}\); also \(y(1) = 1, \quad so \quad y_0 = e^{-0.5887} \approx 0.5550, \quad y \approx 0.5550e^{0.5887t}\).

(d) \(k = \frac{\ln 2}{T} \approx 0.1155, \quad 2 = y(1) \approx y_0e^{0.1155}, \quad y \approx 2e^{-0.1155} \approx 1.7818, \quad y \approx 1.7818e^{0.1155t}\)

14. (a) \(k = \frac{\ln 2}{T} \approx 0.1386, \quad y \approx 10e^{-0.1386t}\)

(b) \(y = 10e^{-0.015t}\)

(c) \(100 = y_0e^{-k}, \quad 1 = y_0e^{-10k}\). Divide: \(e^{9k} = 100, \quad k = \frac{1}{9} \ln 100 \approx 0.5117; \quad y_0 = e^{10k} \approx e^{5.117} \approx 166.83, \quad y = 166.83e^{-0.5117t}\).

(d) \(k = \frac{\ln 2}{T} \approx 0.1386, \quad 10 = y(1) \approx y_0e^{-0.1386}, \quad y_0 \approx 10e^{0.1386} \approx 11.4866, \quad y \approx 11.4866e^{-0.1386t}\)

16. (a) None; the half-life is independent of the initial amount.

(b) \(kT = \ln 2\), so \(T = \text{inversely proportional to} \ k\).

17. (a) \(T = \frac{\ln 2}{k}; \text{and} \ ln 2 \approx 0.6931\). If \(k\) is measured in percent, \(k' = 100k\),

then \(T = \frac{\ln 2}{k'} \approx \frac{69.31}{k'} \approx 70\) yr

(b) \(70\) yr

(c) \(20\) yr

(d) \(7\%\)

18. Let \(y = y_0e^{kt}\) with \(y = y_1\) when \(t = t_1\) and \(y = 3y_1\) when \(t = t_1 + T\); then \(y_0e^{kt_1} = y_1\) (i) and \(y_0e^{k(t_1+T)} = 3y_1\) (ii). Divide (ii) by (i) to get \(e^{kT} = 3\), \(T = \frac{1}{k} \ln 3\).

19. From (11), \(y(t) = y_0e^{-0.000121t}\). If \(0.27 = \frac{y(t)}{y_0} = e^{-0.000121t}\) then \(t = -\frac{\ln 0.27}{0.000121} \approx 10,820\) yr, and

if \(0.30 = \frac{y(t)}{y_0}\) then \(t = -\frac{\ln 0.30}{0.000121} \approx 9950\), or roughly between 9000 B.C. and 8000 B.C.
27. (a) \[ A = 1000e^{0.08}(8) = 1000e^{0.4} \approx 1,491.82 \]

(b) \[ P_e^{0.08}(10) = 10,000, P_e^{0.8} = 10,000, \] \[ P = 10,000e^{-0.8} \approx 4,493.29 \]

(c) From (11), with \( k = r = 0.08 \), \( T = (\ln 2)/0.08 \approx 8.7 \) years.

28. Let \( r \) be the annual interest rate when compounded continuously and \( r_1 \) the effective annual interest rate. Then an amount \( P \) invested at the beginning of the year is worth \( Pe^r = P(1 + r_1) \) at the end of the year, and \( r_1 = e^r - 1 \).

29. (a) \[ \frac{dT}{dt} = -k(T - 21), \quad T(0) = 95, \quad \frac{dT}{T - 21} = -k \, dt, \quad \ln(T - 21) = -kt + C_1, \quad T = 21 + e^{C_1}e^{-kt} = 21 + Ce^{-kt}, \quad 95 = T(0) = 21 + C, \quad C = 74, \quad T = 21 + 74e^{-kt} \]

(b) \[ 85 = T(1) = 21 + 74e^{-k}, \quad \frac{64}{74} = -\ln\left(\frac{32}{37}\right), \quad T = 21 + 74e^{\ln(32/37)} = 21 + 74 \left(\frac{32}{37}\right)^t \]

\[ T = 51 \text{ when } \frac{30}{74} = \left(\frac{32}{37}\right)^t; \quad t = \frac{\ln(30/74)}{\ln(32/37)} \approx 6.22 \text{ min} \]

30. \[ \frac{dT}{dt} = k(70 - T), \quad T(0) = 40; \quad -\ln(70 - T) = kt + C, \quad 70 - T = e^{-kt}e^{-C}, \quad T = 40 \text{ when } t = 0, \quad 30 = e^C, \quad T = 70 - 30e^{-kt}; \quad 52 = T(1) = 70 - 30e^{-k}, \quad k = -\ln\left(\frac{70 - 52}{30}\right) = \ln\left(\frac{5}{3}\right) \approx 0.5, \quad T \approx 70 - 30e^{-0.5t} \]

31. Let \( T \) denote the body temperature of McHam's body at time \( t \), the number of hours elapsed after 10:06 P.M.; then \[ \frac{dT}{dt} = -k(T - 72), \quad \frac{dT}{T - 72} = -k \, dt, \quad \ln(T - 72) = -kt + C, \quad T = 72 + Ce^{-kt}, \quad 77.9 = 72 + e^C, \quad e^C = 5.9, \quad T = 72 + 5.9e^{-kt}, \quad 75.6 = 72 + 5.9e^{-k}, \quad k = -\ln\left(\frac{3.6}{5.9}\right) \approx 0.4940, \]

\[ T = 72 + 5.9e^{-0.4940t} \] McHam's body temperature was last 98.6 \(^\circ\) when \( t = -\frac{\ln(26.6/5.9)}{0.4940} \approx -3.05, \) so around 3 hours and 3 minutes before 10:06; the death took place at approximately 7:03 P.M., while Moore was on stage.

32. If \( T_0 < T_a \) then \[ \frac{dT}{dt} = k(T_a - T) \text{ where } k > 0, \]

If \( T_0 > T_a \) then \[ \frac{dT}{dt} = -k(T - T_a) \text{ where } k > 0; \]

both cases yield \( T(t) = T_a + (T_0 - T_a)e^{-kt} \text{ with } k > 0 \).

33. (a) Both \( y(t) = 0 \) and \( y(t) = L \) are solutions of the logistic equation \[ \frac{dy}{dt} = k \left(1 - \frac{y}{L}\right) y \text{ as both sides of the equation are then zero.} \]

(b) If \( y \) is very small relative to \( L \) then \( y/L \approx 0 \), and the logistic equation becomes \[ \frac{dy}{dt} \approx ky, \] which is a form of the equation for exponential growth.

(c) All the terms on the right-hand-side of the logistic equation are positive, except perhaps \( 1 - \frac{y}{L} \), which is positive if \( y < L \) and negative if \( y > L \).

(d) The rate of change of \( y \) is a function of only one variable, \( y \) itself. The right-hand-side of the differential equation is a quadratic equation in \( y \), which can be thought of as a parabola in \( y \) which opens down and crosses the \( y \)-axis at \( y = 0 \) and \( y = L \). The parabola thus takes its maximum midway between the two \( y \)-intercepts, namely at \( y = L/2 \).

34. (a) Given \[ \frac{dy}{dt} = k \left(1 - \frac{y}{L}\right) y, \] separation of variables yields \[ \frac{1}{L} \left(\frac{1}{y} + \frac{1}{L - y}\right) \, dy = \frac{k}{r} \, dt \text{ so that} \]

\[ \ln y - \ln(L - y) = kt + C. \] The initial condition yields \[ \ln y_0 - \ln(L - y_0) = C. \]