x $1-x$

EXERCISE SET 5.6

1. $f(x) = x^2 - 2,\ f'(x) = 2x,\ x_{n+1} = x_n - \dfrac{x_n^2 - 2}{2x_n}$

   $x_1 = 1,\ x_2 = 1.5,\ x_3 = 1.416666667, \ldots,\ x_5 = x_6 = 1.414213562$

2. $f(x) = x^2 - 5, f'(x) = 2x, x_{n+1} = x_n - \dfrac{x_n^2 - 5}{2x_n}$

   $x_1 = 2, x_2 = 2.25, x_3 = 2.236111111, x_4 = 2.2360679779, x_4 = x_5 = 2.2360679775$

3. $f(x) = x^3 - 6,\ f'(x) = 3x^2,\ x_{n+1} = x_n - \dfrac{x_n^3 - 6}{3x_n^2}$

   $x_1 = 2,\ x_2 = 1.833333333,\ x_3 = 1.817263545, \ldots,\ x_5 = x_6 = 1.817120593$

4. $x^n - a = 0$

5. $f(x) = x^3 - 2x - 2, f'(x) = 3x^2 - 2, x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n - 2}{3x_n^2 - 2}$

   $x_1 = 2, x_2 = 1.8, x_3 = 1.7699481865, x_4 = 1.7692926629, x_5 = x_6 = 1.7692923542$

6. $f(x) = x^3 + x - 1,\ f'(x) = 3x^2 + 1,\ x_{n+1} = x_n - \dfrac{x_n^3 + x_n - 1}{3x_n^2 + 1}$

   $x_1 = 1,\ x_2 = 0.75,\ x_3 = 0.686046512, \ldots,\ x_5 = x_6 = 0.682327804$

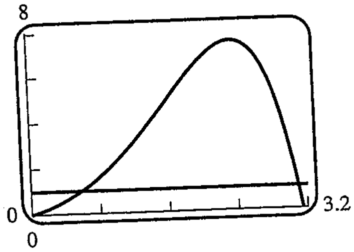
7. $f(x) = x^5 + x^4 - 5,\ f'(x) = 5x^4 + 4x^3,\ x_{n+1} = x_n - \dfrac{x_n^5 + x_n^4 - 5}{5x_n^4 + 4x_n^3}$

   $x_1 = 1,\ x_2 = 1.333333333,\ x_3 = 1.239420573, \ldots, x_6 = x_7 = 1.224439550$

19. The graphs of $y = 1$ and $y = e^x \sin x$ intersect near the points $x = 1$ and $x = 3$. Let $f(x) = 1 - e^x \sin x$, $f'(x) = -e^x(\cos x + \sin x)$, and

$x_{n+1} = x_n + \dfrac{1 - e^x \sin x}{e^x(\cos x + \sin x)}$. If $x_1 = 1$ then

$x_2 = 0.65725814, x_3 = 0.59118311, \ldots, x_5 = x_6 = 0.58853274$, and if $x_1 = 3$ then $x_2 = 3.10759324, x_3 = 3.09649396, \ldots, x_5 = x_6 = 3.09636393$.

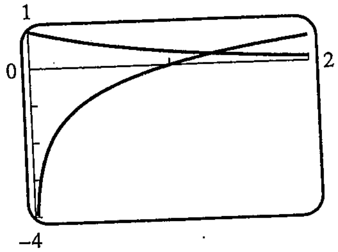


20. The graphs of $y = e^{-x}$ and $y = \ln x$ intersect near $x = 1.3$; let

$f(x) = e^{-x} - \ln x$, $f'(x) = -e^{-x} - 1/x$, $x_1 = 1.3$,

$x_{n+1} = x_n + \dfrac{e^{-x_n} - \ln x_n}{e^{-x_n} + 1/x_n}$, $x_2 = 1.309759929$,

$x_4 = x_5 = 1.309799586$



21. (a) $f(x) = x^2 - a$, $f'(x) = 2x$, $x_{n+1} = x_n - \dfrac{x_n^2 - a}{2x_n} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$

(b) $a = 10$; $x_1 = 3$, $x_2 = 3.166666667$, $x_3 = 3.162280702$, $x_4 = x_5 = 3.162277660$

22. (a) $f(x) = \dfrac{1}{x} - a$, $f'(x) = -\dfrac{1}{x^2}$, $x_{n+1} = x_n(2 - ax_n)$

(b) $a = 17$; $x_1 = 0.05$, $x_2 = 0.0575$, $x_3 = 0.058793750$, $x_5 = x_6 = 0.058823529$

23. $f'(x) = x^3 + 2x - 5$; solve $f'(x) = 0$ to find the critical points. Graph $y = x^3$ and $y = -2x + 5$ to see that they intersect at a point near $x = 1.25$; $f''(x) = 3x^2 + 2$ so $x_{n+1} = x_n - \dfrac{x_n^3 + 2x_n - 5}{3x_n^2 + 2}$.

$x_1 = 1.25, x_2 = 1.3317757009, x_3 = 1.3282755613, x_4 = 1.3282688557, x_5 = 1.3282688557$ so the minimum value of $f(x)$ occurs at $x \approx 1.3282688557$ because $f''(x) > 0$; its value is approximately -4.098859123.

24. From a rough sketch of $y = x \sin x$ we see that the maximum occurs at a point near $x = 2$, which will be a point where $f'(x) = x\cos x + \sin x = 0$. $f''(x) = 2\cos x - x \sin x$ so

$$x_{n+1} = x_n - \frac{x_n \cos x_n + \sin x_n}{2\cos x_n - x_n \sin x_n} = x_n - \frac{x_n + \tan x_n}{2 - x_n \tan x_n}.$$

$x_1 = 2$, $x_2 = 2.029048281$, $x_3 = 2.028757866$, $x_4 = x_5 = 2.028757838$; the maximum value is approximately 1.819705741.

25. A graphing utility shows that there are two inflection points at $x \approx 0.25, -1.25$. These points are the zeros of $f''(x) = (x^4 + 4x^3 + 8x^2 + 4x - 1)\dfrac{e^{-x}}{(x^2+1)^3}$. It is equivalent to find the zeros of $g(x) = x^4 + 4x^3 + 8x^2 + 4x - 1$. One root is $x = -1$ by inspection. Since $g'(x) = 4x^3 + 12x^2 + 16x + 4$, Newton's Method becomes

$$x_{n+1} = x_n - \frac{x_n^4 + 4x_n^3 + 8x_n^2 + 4x_n - 1}{4x_n^3 + 12x_n^2 + 16x_n + 4}$$

With $x_0 = 0.25$, $x_1 = 0.18572695, x_2 = 0.179563312, x_3 = 0.179509029$, $x_4 = x_5 = 0.179509025$. So the points of inflection are at $x \approx 0.18, x = -1$.

35. (a)

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0.5000	−0.7500	0.2917	−1.5685	−0.4654	0.8415	−0.1734	2.7970	1.2197	0.1999

(b) The sequence x_n must diverge, since if it did converge then $f(x) = x^2 + 1 = 0$ would have a solution. It seems the x_n are oscillating back and forth in a quasi-cyclical fashion.

36. (a) $x_{n+1} = x_n$, i.e. the constant sequence x_n is generated.

(b) This is equivalent to $f(x_n) = 0$ as in part (a).

(c) $x_n = x_{n+2} = x_{n+1} - \dfrac{f(x_{n+1})}{f'(x_{n+1})} = x_n - \dfrac{f(x_n)}{f'(x_n)} - \dfrac{f(x_{n+1})}{f'(x_{n+1})}$, so $f(x_{n+1}) = -\dfrac{f'(x_{n+1})}{f'(x_n)}f(x_n)$

37. (a) $|x_{n+1} - x_n| \le |x_{n+1} - c| + |c - x_n| < 1/n + 1/n = 2/n$

(b) The closed interval $[c-1, c+1]$ contains all of the x_n, since $|x_n - c| < 1/n$. Let M be an upper bound for $|f'(x)|$ on $[c-1, c+1]$. Since $x_{n+1} = x_n - f(x_n)/f'(x_n)$ it follows that $|f(x_n)| \le |f'(x_n)||x_{n+1} - x_n| < M|x_{n+1} - x_n| < 2M/n$.

(c) Assume that $f(c) \ne 0$. The sequence x_n converges to c, since $|x_n - c| < 1/n$. By the continuity of f, $f(c) = f(\lim_{n\to+\infty} x_n) = \lim_{n\to+\infty} f(x_n)$.
Let $\epsilon = |f(c)|/2$. Choose N such that $|f(x_n) - f(c)| < \epsilon/2$ for $n > N$. Then $|f(x_n) - f(c)| < |f(c)|/2$ for $n > N$, so $-|f(c)|/2 < f(x_n) - f(c) < |f(c)/2|$.
If $f(c) > 0$ then $f(x_n) > f(c) - |f(c)|/2 = f(c)/2$.
If $f(c) < 0$, then $f(x_n) < f(c) + |f(c)|/2 = -|f(c)|/2$, or $|f(x_n)| > |f(c)|/2$.

(d) From (b) it follows that $\lim_{n\to+\infty} f(x_n) = 0$. From (c) it follows that if $f(c) \ne 0$ then $\lim_{n\to+\infty} f(x_n) \ne 0$, a contradiction. The conclusion, then, is that $f(c) = 0$.

EXERCISE SET 5.7

1. $f(3) = f(5) = 0; f'(x) = 2x - 8, 2c - 8 = 0, c = 4, f'(4) = 0$

2. $f(0) = f(2) = 0,\ f'(x) = 3x^2 - 6x + 2,\ 3c^2 - 6c + 2 = 0;\ c = \dfrac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \sqrt{3}/3$

3. $f(\pi/2) = f(3\pi/2) = 0,\ f'(x) = -\sin x,\ -\sin c = 0,\ c = \pi$

4. $f(-1) = f(3) = 0; f'(1) = 0; f'(x) = 2(1-x)/(4 + 2x - x^2); 2(1-c) = 0, c = 1$

5. $f(0) = f(4) = 0,\ f'(x) = \dfrac{1}{2} - \dfrac{1}{2\sqrt{x}},\ \dfrac{1}{2} - \dfrac{1}{2\sqrt{c}} = 0,\ c = 1$

6. $f(1) = f(3) = 0,\ f'(x) = -\dfrac{2}{x^3} + \dfrac{4}{3x^2},\ -\dfrac{2}{c^3} + \dfrac{4}{3c^2} = 0,\ -6 + 4c = 0,\ c = 3/2$

7. $(f(5) - f(-3))/(5 - (-3)) = 1; f'(x) = 2x - 1; 2c - 1 = 1, c = 1$

8. $f(-1) = -6,\ f(2) = 6,\ f'(x) = 3x^2 + 1,\ 3c^2 + 1 = \dfrac{6 - (-6)}{2 - (-1)} = 4,\ c^2 = 1,\ c = \pm 1$ of which only $c = 1$ is in $(-1, 2)$

9. $f(0) = 1,\ f(3) = 2,\ f'(x) = \dfrac{1}{2\sqrt{x+1}},\ \dfrac{1}{2\sqrt{c+1}} = \dfrac{2-1}{3-0} = \dfrac{1}{3},\ \sqrt{c+1} = 3/2,\ c + 1 = 9/4,\ c = 5/4$

10. $f(4) = 15/4, f(3) = 8/3$, solve $f'(c) = (15/4 - 8/3)/1 = 13/12$; $f'(x) = 1 + 1/x^2, f'(c) = 1 + 1/c^2 = 13/12, c^2 = 12, c = \pm 2\sqrt{3}$, but $-2\sqrt{3}$ is not in the interval, so $c = 2\sqrt{3}$.