EXERCISE SET 5.6

1. \( f(x) = x^2 - 2, \ f'(x) = 2x, \ x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} \)
   \( x_1 = 1, \ x_2 = 1.5, \ x_3 = 1.416666667, \ldots, \ x_5 = x_6 = 1.414213562 \)

2. \( f(x) = x^2 - 5, \ f'(x) = 2x, \ x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n} \)
   \( x_1 = 2, \ x_2 = 2.25, \ x_3 = 2.236111111, \ x_4 = 2.2360679779, \ x_5 = x_6 = 2.2360679775 \)

3. \( f(x) = x^3 - 6, \ f'(x) = 3x^2, \ x_{n+1} = x_n - \frac{x_n^3 - 6}{3x_n^2} \)
   \( x_1 = 2, \ x_2 = 1.833333333, \ x_3 = 1.817263545, \ldots, \ x_5 = x_6 = 1.817120593 \)

4. \( x^n - \alpha = 0 \)

5. \( f(x) = x^3 - 2x - 2, \ f'(x) = 3x^2 - 2, \ x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 2}{3x_n^2 - 2} \)
   \( x_1 = 2, \ x_2 = 1.8, \ x_3 = 1.7699481865, \ x_4 = 1.7692926629, \ x_5 = x_6 = 1.7692923542 \)

6. \( f(x) = x^3 + x - 1, \ f'(x) = 3x^2 + 1, \ x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1} \)
   \( x_1 = 1, \ x_2 = 0.75, \ x_3 = 0.686046512, \ldots, \ x_5 = x_6 = 0.682327804 \)

7. \( f(x) = x^5 + x^4 - 5, \ f'(x) = 5x^4 + 4x^3, \ x_{n+1} = x_n - \frac{x_n^5 + x_n^4 - 5}{5x_n^4 + 4x_n^3} \)
   \( x_1 = 1, \ x_2 = 1.333333333, \ x_3 = 1.239420573, \ldots, \ x_5 = x_7 = 1.224439550 \)
19. The graphs of \( y = 1 \) and \( y = e^x \sin x \) intersect near the points \( x = 1 \) and \( x = 3 \). Let \( f(x) = 1 - e^x \sin x, f'(x) = -e^x(\cos x + \sin x) \), and 
\[
x_{n+1} = x_n + \frac{1 - e^x \sin x}{e^x(\cos x + \sin x)}. \quad \text{If } x_1 = 1 \text{ then}
\]
x_2 = 0.65725514, x_3 = 0.59118311, \ldots, x_5 = x_6 = 0.58855374, and if \( x_1 = 3 \) then \( x_2 = 3.10759324, x_3 = 3.09649396, \ldots, x_5 = x_6 = 3.09636393 \). 

20. The graphs of \( y = e^{-x} \) and \( y = \ln x \) intersect near \( x = 1.3 \); let 
\[
f(x) = e^{-x} - \ln x, \quad f'(x) = -e^{-x} - 1/x, \quad x_1 = 1.3,
\]
x_2 = x_3 = x_4 = x_5 = 1.309799586

21. (a) \( f(x) = x^2 - a, \quad f'(x) = 2x, \quad x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) \)
(b) \( a = 10; x_1 = 3, x_2 = 3.16666667, x_3 = 3.162280702, x_4 = x_5 = 3.162277660 \)

22 (a) \( f(x) = \frac{1}{x} - a, \quad f'(x) = -\frac{1}{x^2}, \quad x_{n+1} = x_n(2 - ax_n) \)
(b) \( a = 17; x_1 = 0.05, x_2 = 0.0575, x_3 = 0.058793750, x_5 = x_6 = 0.058823529 \)

23. \( f'(x) = x^3 + 2x - 5 \); solve \( f'(x) = 0 \) to find the critical points. Graph \( y = x^3 \) and \( y = -2x + 5 \) to see that they intersect at a point near \( x = 1.25 \); \( f''(x) = 3x^2 + 2 \) so \( x_{n+1} = x_n - \frac{x_n^3 + 2x_n - 5}{3x_n^2 + 2} \).

\[ x_1 = 1.25, x_2 = 1.3317757009, x_3 = 1.3282755613, x_4 = 1.3282688557, x_5 = 1.3282688557 \]
so the minimum value of \( f(x) \) occurs at \( x \approx 1.3282688557 \) because \( f''(x) > 0 \); its value is approximately \(-4.098859123 \).

24. From a rough sketch of \( y = x \sin x \) we see that the maximum occurs at a point near \( x = 2 \), which will be a point where \( f'(x) = x \cos x + \sin x = 0 \). \( f''(x) = 2 \cos x - x \sin x \) so
\[
x_{n+1} = x_n - \frac{x_n \cos x_n + \sin x_n}{2 \cos x_n - x_n \sin x_n} = x_n - \frac{x_n + \tan x_n}{2 - x_n \tan x_n}
\]
x_1 = 2, x_2 = 2.029042581, x_3 = 2.028757866, x_4 = x_5 = 2.028757838; the maximum value is approximately 1.819705741.

25. A graphing utility shows that there are two inflection points at \( x \approx 0.25, -1.25 \). These points are the zeros of \( f''(x) = (x^4 + 4x^3 + 8x^2 + 4x - 1) \frac{e^{-x}}{(x^2 + 1)^3} \). It is equivalent to find the zeros of \( g(x) = x^4 + 4x^3 + 8x^2 + 4x - 1 \). One root is \( x = -1 \) by inspection. Since \( g'(x) = 4x^3 + 12x^2 + 16x + 4 \),

Newton's Method becomes 
\[
x_{n+1} = x_n - \frac{x_n^4 + 4x_n^3 + 8x_n^2 + 4x_n - 1}{4x_n^3 + 12x_n^2 + 16x_n + 4}
\]

With \( x_0 = 0.25, x_1 = -0.48572695, x_2 = 0.179563312, x_3 = 0.179509029, x_4 = x_5 = 0.179509025 \). So the points of inflection are at \( x \approx 0.18, x = -1 \).
Exercise Set 5.7

35. (a) The sequence $x_n$ must diverge, since if it did converge then $f(x) = x^2 + 1 = 0$ would have a solution. It seems the sequence $x_n$ is oscillating back and forth in a quasi-cyclical fashion.

(b) This is equivalent to $f(x_n) = 0$ as in part (a).

(c) $x_n = x_{n+2} - x_{n+1} - \frac{f'(x_{n+1})}{f''(x_{n+1})} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_{n+1})}{f'(x_{n+1})}$. So $f(x_{n+1}) = -\frac{f'(x_{n+1})}{f'(x_n)} f(x_n)$

37. (a) $|x_{n+1} - x_n| \leq |x_{n+1} - c| + |c - x_n| < 1/n + 1/n = 2/n$

(b) The closed interval $[c-1, c+1]$ contains all of the $x_n$, since $|x_n - c| < 1/n$. Let $M$ be an upper bound for $|f''(x)|$ on $[c-1, c+1]$. Since $x_{n+1} = x_n - f(x_n)/f'(x_n)$ it follows that $|f(x_n)| \leq |f'(x_n)||x_{n+1} - x_n| < M|x_{n+1} - x_n| < 2M/n$. Assume that $f(c) \neq 0$. The sequence $x_n$ converges to $c$, since $|x_n - c| < 1/n$. By the continuity of $f$, $f(c) = f(\lim_{n \to +\infty} x_n) = \lim_{n \to +\infty} f(x_n)$. Let $\epsilon = |f(c)|/2$. Choose $N$ such that $|f(x_n) - f(c)| < \epsilon/2$ for $n > N$. Then $|f(x_n) - f(c)| < |f(c)|/2$ for $n > N$, so $-\epsilon/2 < f(x_n) - f(c) < \epsilon/2$.

If $f(c) > 0$ then $f(x_n) > f(c) - |f(c)|/2 = f(c)/2$. If $f(c) < 0$, then $f(x_n) < f(c) + |f(c)|/2 = -|f(c)|/2$, or $|f(x_n)| > |f(c)|/2$.

(d) From (b) it follows that $\lim_{n \to +\infty} f(x_n) = 0$. From (c) it follows that if $f(c) \neq 0$ then $\lim_{n \to +\infty} f(x_n) \neq 0$, a contradiction. The conclusion, then, is that $f(c) = 0$.

EXERCISE SET 5.7

1. $f(3) = f(5) = 0; f'(x) = 2x - 8, 2c - 8 = 0, c = 4, f'(4) = 0$

2. $f(0) = f(2) = 0, f'(x) = 3x^2 - 6x + 2, 3c^2 - 6c + 2 = 0; c = \frac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \sqrt{3}/3$

3. $f(\pi/2) = f(3\pi/2) = 0, f'(x) = -\sin x, -\sin c = 0, c = \pi$

4. $f(-1) = f(3) = 0; f'(1) = 0; f'(x) = 2(1 - x)/(4 + 2x - x^2); 2(1 - c) = 0, c = 1$

5. $f(0) = f(4) = 0, f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{x}} = \frac{1}{2} - \frac{1}{2\sqrt{c}} = 0, c = 1$

6. $f(1) = f(3) = 0, f'(x) = -\frac{2}{x^3} + \frac{4}{3x^2} = \frac{4}{3c^2} = 0, -6 + 4c = 0, c = 3/2$

7. $(f(5) - f(-3))/(5 - (-3)) = 1; f'(x) = 2x - 1, 2c - 1 = 1, c = 1$

8. $f(-1) = -6, f(2) = 6, f'(x) = 3x^2 + 1, 3c^2 + 1 = 6 - \frac{6}{2} = 1, c = 1/2, c = \pm 1$ of which only $c = 1$ is in $(-1, 2)$

9. $f(0) = 1, f(3) = 2, f'(x) = \frac{1}{2\sqrt{x + 1}}, \frac{1}{2\sqrt{c + 1}} = 2 + 1 \frac{1}{3 - 0} = \frac{1}{3}, \sqrt{c + 1} = 3/2, c + 1 = 9/4, c = 5/4$

10. $f(4) = 15/4, f(3) = 8/3$, solve $f'(c) = (15/4 - 8/3)/1 = 13/12; f'(x) = x + 1/x^2, f'(c) = 1 + 1/c^2 = 13/12, c^2 = 12, c = \pm2\sqrt{3}$, but $-2\sqrt{3}$ is not in the interval, so $c = 2\sqrt{3}$.