EXERCISE SET 5.5

1. If $y = x + 1/x$ for $1/2 \leq x \leq 3/2$ then $dy/dx = 1 - 1/x^2 = (x^2 - 1)/x^2$, $dy/dx = 0$ when $x = 1$. If $x = 1/2, 1, 3/2$ then $y = 5/2, 2, 13/6$ so

   (a) y is as small as possible when $x = 1$.

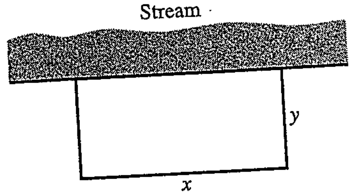
   (b) y is as large as possible when $x = 1/2$.

2. Let x and y be nonnegative numbers and z the sum of their squares, then $z = x^2 + y^2$. But $x + y = 1$, $y = 1 - x$ so $z = x^2 + (1-x)^2 = 2x^2 - 2x + 1$ for $0 \leq x \leq 1$. $dz/dx = 4x - 2$, $dz/dx = 0$ when $x = 1/2$. If $x = 0, 1/2, 1$ then $z = 1, 1/2, 1$ so

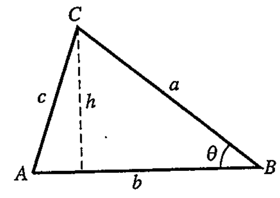
   (a) z is as large as possible when one number is 0 and the other is 1.

   (b) z is as small as possible when both numbers are 1/2.

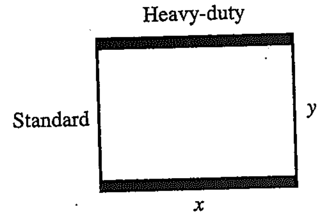
3. $A = xy$ where $x + 2y = 1000$ so $y = 500 - x/2$ and $A = 500x - x^2/2$ for x in $[0, 1000]$; $dA/dx = 500 - x$, $dA/dx = 0$ when $x = 500$. If $x = 0$ or 1000 then $A = 0$, if $x = 500$ then $A = 125{,}000$ so the area is maximum when $x = 500$ ft and $y = 500 - 500/2 = 250$ ft.



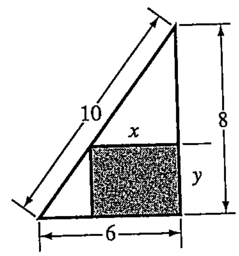
4. The triangle in the figure is determined by the two sides of length a and b and the angle θ. Suppose that $a + b = 1000$. Then $h = a \sin\theta$ and area $= Q = hb/2 = ab(\sin\theta)/2$. This expression is maximized by choosing $\theta = \pi/2$ and thus $Q = ab/2 = a(1000 - a)/2$, which is maximized by $a = 500$. Thus the maximal area is obtained by a right isosceles triangle, where the angle between two sides of 500 is the right angle.



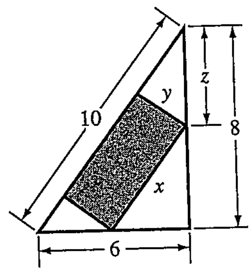
5. Let x and y be the dimensions shown in the figure and A the area, then $A = xy$ subject to the cost condition $3(2x) + 2(2y) = 6000$, or $y = 1500 - 3x/2$. Thus $A = x(1500 - 3x/2) = 1500x - 3x^2/2$ for x in $[0, 1000]$. $dA/dx = 1500 - 3x$, $dA/dx = 0$ when $x = 500$. If $x = 0$ or 1000 then $A = 0$, if $x = 500$ then $A = 375{,}000$ so the area is greatest when $x = 500$ ft and (from $y = 1500 - 3x/2$) when $y = 750$ ft.



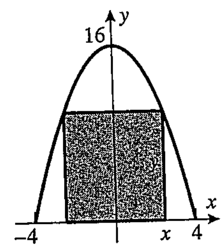
6. Let x and y be the dimensions shown in the figure and A the area of the rectangle, then $A = xy$ and, by similar triangles, $x/6 = (8 - y)/8$, $y = 8 - 4x/3$ so $A = x(8 - 4x/3) = 8x - 4x^2/3$ for x in $[0, 6]$. $dA/dx = 8 - 8x/3$, $dA/dx = 0$ when $x = 3$. If $x = 0, 3, 6$ then $A = 0, 12, 0$ so the area is greatest when $x = 3$ in and (from $y = 8 - 4x/3$) $y = 4$ in.



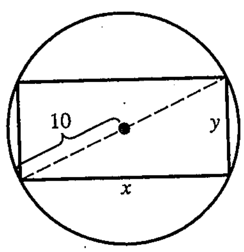
7. Let x, y, and z be as shown in the figure and A the area of the rectangle, then $A = xy$ and, by similar triangles, $z/10 = y/6$, $z = 5y/3$; also $x/10 = (8 - z)/8 = (8 - 5y/3)/8$ thus $y = 24/5 - 12x/25$ so $A = x(24/5 - 12x/25) = 24x/5 - 12x^2/25$ for x in $[0, 10]$. $dA/dx = 24/5 - 24x/25$, $dA/dx = 0$ when $x = 5$. If $x = 0, 5, 10$ then $A = 0, 12, 0$ so the area is greatest when $x = 5$ in. and $y = 12/5$ in.



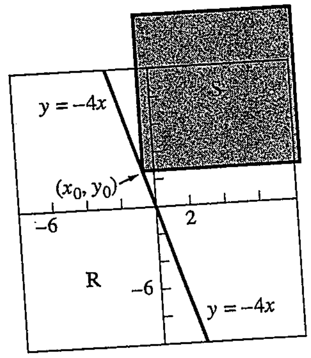
8. $A = (2x)y = 2xy$ where $y = 16 - x^2$ so $A = 32x - 2x^3$ for $0 \le x \le 4$; $dA/dx = 32 - 6x^2$, $dA/dx = 0$ when $x = 4/\sqrt{3}$. If $x = 0, 4/\sqrt{3}, 4$ then $A = 0, 256/(3\sqrt{3}), 0$ so the area is largest when $x = 4/\sqrt{3}$ and $y = 32/3$. The dimensions of the rectangle with largest area are $8/\sqrt{3}$ by $32/3$.



9. $A = xy$ where $x^2 + y^2 = 20^2 = 400$ so $y = \sqrt{400 - x^2}$ and $A = x\sqrt{400 - x^2}$ for $0 \le x \le 20$; $dA/dx = 2(200 - x^2)/\sqrt{400 - x^2}$, $dA/dx = 0$ when $x = \sqrt{200} = 10\sqrt{2}$. If $x = 0, 10\sqrt{2}, 20$ then $A = 0, 200, 0$ so the area is maximum when $x = 10\sqrt{2}$ and $y = \sqrt{400 - 200} = 10\sqrt{2}$.



10. Let R denote the rectangle with corners $(\pm 8, \pm 10)$. Suppose the lower left corner of the square S is at the point $(x_0, y_0) = (x_0, -4x_0)$. Let Q denote the desired region.



It is clear that Q will be a rectangle, and that its left and bottom sides are subsets of the left and bottom sides of S. The right and top edges of S will be subsets of the sides of R (if S is big enough).

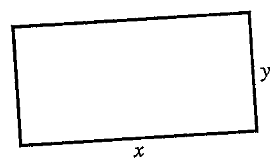
From the drawing it is evident that the area of Q is $(8-x_0)(10+4x_0) = -4x_0^2 + 22x_0 + 80$. This function is maximized when $x_0 = 11/4$, for which the area of Q is $441/4$.
The maximum possible area for Q is $441/4$, taken when $x_0 = 11/4$.

11. Let $x =$ length of each side that uses the \$1 per foot fencing,
$y =$ length of each side that uses the \$2 per foot fencing.
The cost is $C = (1)(2x) + (2)(2y) = 2x + 4y$, but $A = xy = 3200$ thus $y = 3200/x$ so

$$C = 2x + 12800/x \text{ for } x > 0,$$
$$dC/dx = 2 - 12800/x^2,\ dC/dx = 0 \text{ when } x = 80,\ d^2C/dx^2 > 0 \text{ so}$$

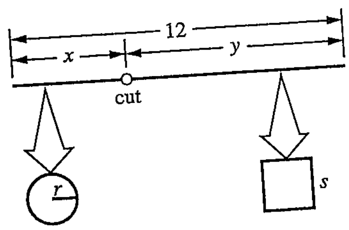
C is least when $x = 80$, $y = 40$.

12. $A = xy$ where $2x + 2y = p$ so $y = p/2 - x$ and $A = px/2 - x^2$ for x in $[0, p/2]$; $dA/dx = p/2 - 2x$, $dA/dx = 0$ when $x = p/4$. If $x = 0$ or $p/2$ then $A = 0$, if $x = p/4$ then $A = p^2/16$ so the area is maximum when $x = p/4$ and $y = p/2 - p/4 = p/4$, which is a square.



13. Let x and y be the dimensions of a rectangle; the perimeter is $p = 2x + 2y$. But $A = xy$ thus $y = A/x$ so $p = 2x + 2A/x$ for $x > 0$, $dp/dx = 2 - 2A/x^2 = 2(x^2 - A)/x^2$, $dp/dx = 0$ when $x = \sqrt{A}$, $d^2p/dx^2 = 4A/x^3 > 0$ if $x > 0$ so p is a minimum when $x = \sqrt{A}$ and $y = \sqrt{A}$ and thus the rectangle is a square.

14. With x, y, r, and s as shown in the figure, the sum of the enclosed areas is $A = \pi r^2 + s^2$ where $r = \dfrac{x}{2\pi}$ and $s = \dfrac{y}{4}$ because x is the circumference of the circle and y is the perimeter of the square, thus $A = \dfrac{x^2}{4\pi} + \dfrac{y^2}{16}$. But $x + y = 12$, so $y = 12 - x$ and



$$A = \frac{x^2}{4\pi} + \frac{(12-x)^2}{16} = \frac{\pi+4}{16\pi}x^2 - \frac{3}{2}x + 9 \text{ for } 0 \le x \le 12.$$

$$\frac{dA}{dx} = \frac{\pi+4}{8\pi}x - \frac{3}{2},\ \frac{dA}{dx} = 0 \text{ when } x = \frac{12\pi}{\pi+4}. \text{ If } x = 0, \frac{12\pi}{\pi+4}, 12$$

then $A = 9, \dfrac{36}{\pi+4}, \dfrac{36}{\pi}$ so the sum of the enclosed areas is

(a) a maximum when $x = 12$ in. (when all of the wire is used for the circle)
(b) a minimum when $x = 12\pi/(\pi+4)$ in.

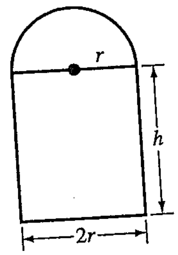
15. (a) $\dfrac{dN}{dt} = 250(20-t)e^{-t/20} = 0$ at $t = 20$, $N(0) = 125{,}000$, $N(20) \approx 161{,}788$, and $N(100) \approx 128{,}369$; the absolute maximum is $N = 161788$ at $t = 20$, the absolute minimum is $N = 125{,}000$ at $t = 0$.

(b) The absolute minimum of $\dfrac{dN}{dt}$ occurs when $\dfrac{d^2N}{dt^2} = 12.5(t-40)e^{-t/20} = 0$, $t = 40$.

16. The area of the window is $A = 2rh + \pi r^2/2$, the perimeter is $p = 2r + 2h + \pi r$ thus $h = \frac{1}{2}[p - (2+\pi)r]$ so

$$\begin{aligned} A &= r[p-(2+\pi)r] + \pi r^2/2 \\ &= pr - (2+\pi/2)r^2 \text{ for } 0 \le r \le p/(2+\pi), \end{aligned}$$

$dA/dr = p - (4+\pi)r$, $dA/dr = 0$ when $r = p/(4+\pi)$ and $d^2A/dr^2 < 0$, so A is maximum when $r = p/(4+\pi)$.



17. Let the box have dimensions x, x, y, with $y \ge x$. The constraint is $4x + y \le 108$, and the volume $V = x^2y$. If we take $y = 108 - 4x$ then $V = x^2(108-4x)$ and $dV/dx = 12x(-x+18)$ with roots $x = 0, 18$. The maximum value of V occurs at $x = 18, y = 36$ with $V = 11,664$ in^2.

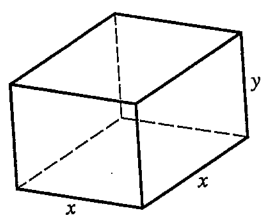
18. Let the box have dimensions x, x, y with $x \ge y$. The constraint is $x+2(x+y) \le 108$, and the volume $V = x^2y$. Take $x = (108-2y)/3 = 36 - 2y/3$, $V = y(36-2y/3)^2$, $dV/dy = (4/3)y^2 - 96y + 1296$ with roots $y = 18, 54$. Then $d^2V/dy^2 = (8/3)y - 96$ is negative for $y = 18$, so by the second derivative test, V has a maximum of $10,368$ in^2 at $y = 18, x = 24$.

19. Let x be the length of each side of a square, then $V = x(3-2x)(8-2x) = 4x^3 - 22x^2 + 24x$ for $0 \le x \le 3/2$; $dV/dx = 12x^2 - 44x + 24 = 4(3x-2)(x-3)$, $dV/dx = 0$ when $x = 2/3$ for $0 < x < 3/2$. If $x = 0, 2/3, 3/2$ then $V = 0, 200/27, 0$ so the maximum volume is $200/27$ ft^3.

20. Let x = length of each edge of base, y = height. The cost is $C =$ (cost of top and bottom) + (cost of sides) $= (2)(2x^2) + (3)(4xy) = 4x^2 + 12xy$, but $V = x^2y = 2250$ thus $y = 2250/x^2$ so $C = 4x^2 + 27000/x$ for $x > 0$, $dC/dx = 8x - 27000/x^2$, $dC/dx = 0$ when $x = \sqrt[3]{3375} = 15$, $d^2C/dx^2 > 0$ so C is least when $x = 15$, $y = 10$.

21. Let x = length of each edge of base, y = height, $k = \$/\text{cm}^2$ for the sides. The cost is $C = (2k)(2x^2) + (k)(4xy) = 4k(x^2 + xy)$, but $V = x^2y = 2000$ thus $y = 2000/x^2$ so $C = 4k(x^2 + 2000/x)$ for $x > 0$, $dC/dx = 4k(2x - 2000/x^2)$, $dC/dx = 0$ when $x = \sqrt[3]{1000} = 10$, $d^2C/dx^2 > 0$ so C is least when $x = 10$, $y = 20$.

22. Let x and y be the dimensions shown in the figure and V the volume, then $V = x^2y$. The amount of material is to be 1000 ft^2, thus (area of base) + (area of sides) $= 1000$, $x^2 + 4xy = 1000$, $y = \dfrac{1000 - x^2}{4x}$ so $V = x^2\dfrac{1000-x^2}{4x} = \dfrac{1}{4}(1000x - x^3)$ for $0 < x \le 10\sqrt{10}$. $\dfrac{dV}{dx} = \dfrac{1}{4}(1000 - 3x^2)$, $\dfrac{dV}{dx} = 0$



when $x = \sqrt{1000/3} = 10\sqrt{10/3}$.

If $x = 0, 10\sqrt{10/3}, 10\sqrt{10}$ then $V = 0, \dfrac{5000}{3}\sqrt{10/3}, 0$;

the volume is greatest for $x = 10\sqrt{10/3}$ ft and $y = 5\sqrt{10/3}$ ft.

23. Let x = height and width, y = length. The surface area is $S = 2x^2 + 3xy$ where $x^2y = V$, so $y = V/x^2$ and $S = 2x^2 + 3V/x$ for $x > 0$; $dS/dx = 4x - 3V/x^2$, $dS/dx = 0$ when $x = \sqrt[3]{3V/4}$, $d^2S/dx^2 > 0$ so S is minimum when $x = \sqrt[3]{\dfrac{3V}{4}}$, $y = \dfrac{4}{3}\sqrt[3]{\dfrac{3V}{4}}$.

24. Let r and h be the dimensions shown in the figure, then the volume of the inscribed cylinder is $V = \pi r^2 h$. But

$r^2 + \left(\dfrac{h}{2}\right)^2 = R^2$ thus $r^2 = R^2 - \dfrac{h^2}{4}$

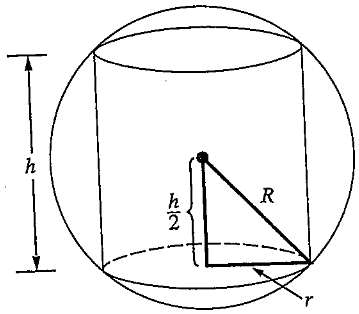
so $V = \pi\left(R^2 - \dfrac{h^2}{4}\right)h = \pi\left(R^2h - \dfrac{h^3}{4}\right)$

for $0 \le h \le 2R$. $\dfrac{dV}{dh} = \pi\left(R^2 - \dfrac{3}{4}h^2\right)$, $\dfrac{dV}{dh} = 0$

when $h = 2R/\sqrt{3}$. If $h = 0, 2R/\sqrt{3}, 2R$

then $V = 0, \dfrac{4\pi}{3\sqrt{3}}R^3, 0$ so the volume is largest

when $h = 2R/\sqrt{3}$ and $r = \sqrt{2/3}R$.



25. Let r and h be the dimensions shown in the figure, then the surface area is $S = 2\pi rh + 2\pi r^2$.

But $r^2 + \left(\dfrac{h}{2}\right)^2 = R^2$ thus $h = 2\sqrt{R^2 - r^2}$ so

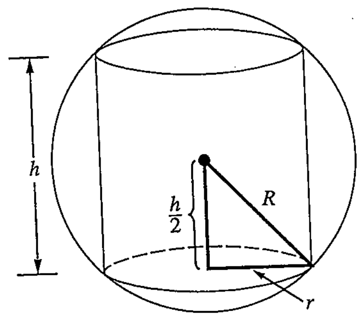
$S = 4\pi r\sqrt{R^2 - r^2} + 2\pi r^2$ for $0 \le r \le R$,

$\dfrac{dS}{dr} = \dfrac{4\pi(R^2 - 2r^2)}{\sqrt{R^2 - r^2}} + 4\pi r$; $\dfrac{dS}{dr} = 0$ when

$$\frac{R^2 - 2r^2}{\sqrt{R^2 - r^2}} = -r \qquad (1)$$

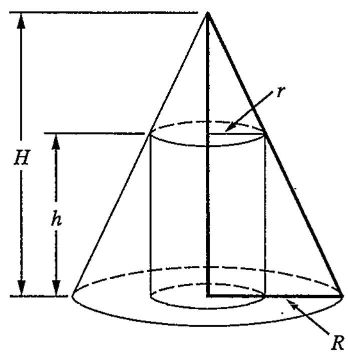
$$R^2 - 2r^2 = -r\sqrt{R^2 - r^2}$$

$$R^4 - 4R^2r^2 + 4r^4 = r^2(R^2 - r^2)$$
$$5r^4 - 5R^2r^2 + R^4 = 0$$



and using the quadratic formula $r^2 = \dfrac{5R^2 \pm \sqrt{25R^4 - 20R^4}}{10} = \dfrac{5 \pm \sqrt{5}}{10}R^2$, $r = \sqrt{\dfrac{5 \pm \sqrt{5}}{10}}R$, of which only $r = \sqrt{\dfrac{5 + \sqrt{5}}{10}}R$ satisfies (1). If $r = 0, \sqrt{\dfrac{5+\sqrt{5}}{10}}R, 0$ then $S = 0, (5 + \sqrt{5})\pi R^2, 2\pi R^2$ so the surface area is greatest when $r = \sqrt{\dfrac{5 + \sqrt{5}}{10}}R$ and, from $h = 2\sqrt{R^2 - r^2}$, $h = 2\sqrt{\dfrac{5 - \sqrt{5}}{10}}R$.

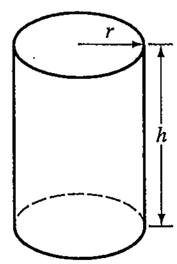
26. Let R and H be the radius and height of the cone, and r and h the radius and height of the cylinder (see figure), then the volume of the cylinder is $V = \pi r^2 h$. By similar triangles (see figure) $\dfrac{H-h}{H} = \dfrac{r}{R}$ thus $h = \dfrac{H}{R}(R-r)$ so $V = \pi\dfrac{H}{R}(R-r)r^2 = \pi\dfrac{H}{R}(Rr^2 - r^3)$ for $0 \le r \le R$. $\dfrac{dV}{dr} = \pi\dfrac{H}{R}(2Rr - 3r^2) = \pi\dfrac{H}{R}r(2R-3r)$, $\dfrac{dV}{dr} = 0$ for $0 < r < R$ when $r = 2R/3$. If $r = 0, 2R/3, R$ then $V = 0, 4\pi R^2H/27, 0$ so the maximum volume is $\dfrac{4\pi R^2 H}{27} = \dfrac{4}{9}\dfrac{1}{3}\pi R^2 H = \dfrac{4}{9}\cdot$ (volume of cone).



27. From (13), $S = 2\pi r^2 + 2\pi rh$. But $V = \pi r^2 h$ thus $h = V/(\pi r^2)$ and so $S = 2\pi r^2 + 2V/r$ for $r > 0$. $dS/dr = 4\pi r - 2V/r^2$, $dS/dr = 0$ if $r = \sqrt[3]{V/(2\pi)}$. Since $d^2S/dr^2 = 4\pi + 4V/r^3 > 0$, the minimum surface area is achieved when $r = \sqrt[3]{V/2\pi}$ and so $h = V/(\pi r^2) = [V/(\pi r^3)]r = 2r$.

28. $V = \pi r^2 h$ where $S = 2\pi r^2 + 2\pi rh$ so $h = \dfrac{S - 2\pi r^2}{2\pi r}$, $V = \dfrac{1}{2}(Sr - 2\pi r^3)$ for $r > 0$.
$\dfrac{dV}{dr} = \dfrac{1}{2}(S - 6\pi r^2) = 0$ if $r = \sqrt{S/(6\pi)}$, $\dfrac{d^2V}{dr^2} = -6\pi r < 0$ so V is maximum when $r = \sqrt{S/(6\pi)}$ and $h = \dfrac{S-2\pi r^2}{2\pi r} = \dfrac{S-2\pi r^2}{2\pi r^2}r = \dfrac{S - S/3}{S/3}r = 2r$, thus the height is equal to the diameter of the base.

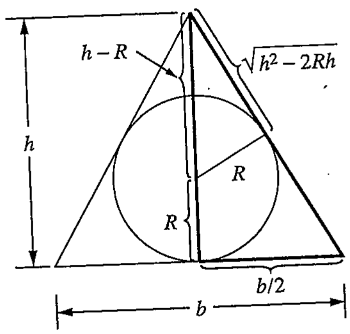
29. The surface area is $S = \pi r^2 + 2\pi rh$ where $V = \pi r^2 h = 500$ so $h = 500/(\pi r^2)$ and $S = \pi r^2 + 1000/r$ for $r > 0$;
$dS/dr = 2\pi r - 1000/r^2 = (2\pi r^3 - 1000)/r^2$,
$dS/dr = 0$ when $r = \sqrt[3]{500/\pi}$, $d^2S/dr^2 > 0$ for $r > 0$ so S is minimum when $r = \sqrt[3]{500/\pi}$ cm and
$$h = \frac{500}{\pi r^2} = \frac{500}{\pi}\left(\frac{\pi}{500}\right)^{2/3} = \sqrt[3]{500/\pi}\text{ cm}$$



30. The total area of material used is
$A = A_{\text{top}} + A_{\text{bottom}} + A_{\text{side}} = (2r)^2 + (2r)^2 + 2\pi rh = 8r^2 + 2\pi rh$.
The volume is $V = \pi r^2 h$ thus $h = V/(\pi r^2)$ so $A = 8r^2 + 2V/r$ for $r > 0$,
$dA/dr = 16r - 2V/r^2 = 2(8r^3 - V)/r^2$, $dA/dr = 0$ when $r = \sqrt[3]{V}/2$. This is the only critical point, $d^2A/dr^2 > 0$ there so the least material is used when $r = \sqrt[3]{V}/2$, $\dfrac{r}{h} = \dfrac{r}{V/(\pi r^2)} = \dfrac{\pi}{V}r^3$ and, for $r = \sqrt[3]{V}/2$, $\dfrac{r}{h} = \dfrac{\pi}{V}\dfrac{V}{8} = \dfrac{\pi}{8}$.

31. Let x be the length of each side of the squares and y the height of the frame, then the volume is $V = x^2 y$. The total length of the wire is L thus $8x + 4y = L$, $y = (L - 8x)/4$ so $V = x^2(L - 8x)/4 = (Lx^2 - 8x^3)/4$ for $0 \le x \le L/8$. $dV/dx = (2Lx - 24x^2)/4$, $dV/dx = 0$ for $0 < x < L/8$ when $x = L/12$. If $x = 0, L/12, L/8$ then $V = 0, L^3/1728, 0$ so the volume is greatest when $x = L/12$ and $y = L/12$.

36. The area of the triangle is $A = \frac{1}{2}hb$. By similar triangles (see figure) $\frac{b/2}{h} = \frac{R}{\sqrt{h^2-2Rh}}$, $b = \frac{2Rh}{\sqrt{h^2-2Rh}}$ so $A = \frac{Rh^2}{\sqrt{h^2-2Rh}}$ for $h > 2R$, $\frac{dA}{dh} = \frac{Rh^2(h-3R)}{(h^2-2Rh)^{3/2}}$, $\frac{dA}{dh} = 0$ for $h > 2R$ when $h = 3R$, by the first derivative test A is minimum when $h = 3R$. If $h = 3R$ then $b = 2\sqrt{3}R$ (the triangle is equilateral).

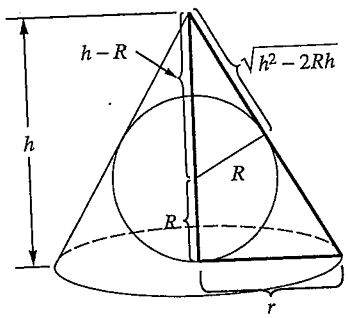


37. The volume of the cone is $V = \frac{1}{3}\pi r^2 h$. By similar triangles (see figure) $\frac{r}{h} = \frac{R}{\sqrt{h^2-2Rh}}$, $r = \frac{Rh}{\sqrt{h^2-2Rh}}$

so $V = \frac{1}{3}\pi R^2 \frac{h^3}{h^2-2Rh} = \frac{1}{3}\pi R^2 \frac{h^2}{h-2R}$ for $h > 2R$,

$\frac{dV}{dh} = \frac{1}{3}\pi R^2 \frac{h(h-4R)}{(h-2R)^2}$, $\frac{dV}{dh} = 0$ for $h > 2R$ when

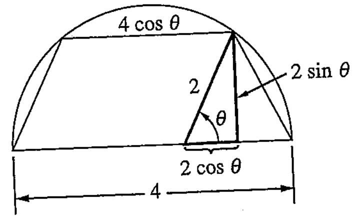
$h = 4R$, by the first derivative test V is minimum when $h = 4R$. If $h = 4R$ then $r = \sqrt{2}R$.



38. The area is (see figure)

$A = \frac{1}{2}(2\sin\theta)(4+4\cos\theta)$
$= 4(\sin\theta + \sin\theta\cos\theta)$
for $0 \le \theta \le \pi/2$;
$dA/d\theta = 4(\cos\theta - \sin^2\theta + \cos^2\theta)$
$= 4(\cos\theta - [1-\cos^2\theta] + \cos^2\theta)$
$= 4(2\cos^2\theta + \cos\theta - 1)$
$= 4(2\cos\theta - 1)(\cos\theta + 1)$

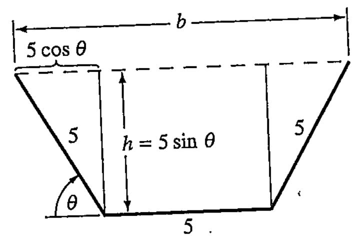
$dA/d\theta = 0$ when $\theta = \pi/3$ for $0 < \theta < \pi/2$. If $\theta = 0, \pi/3, \pi/2$ then $A = 0, 3\sqrt{3}, 4$ so the maximum area is $3\sqrt{3}$.



39. Let b and h be the dimensions shown in the figure, then the cross-sectional area is $A = \frac{1}{2}h(5+b)$. But $h = 5\sin\theta$ and $b = 5 + 2(5\cos\theta) = 5 + 10\cos\theta$ so $A = \frac{5}{2}\sin\theta(10 + 10\cos\theta) = 25\sin\theta(1+\cos\theta)$ for $0 \le \theta \le \pi/2$.

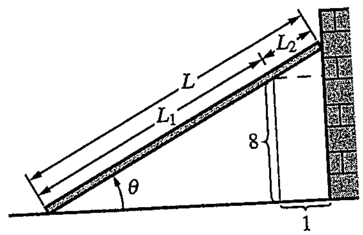
$dA/d\theta = -25\sin^2\theta + 25\cos\theta(1+\cos\theta)$
$= 25(-\sin^2\theta + \cos\theta + \cos^2\theta)$
$= 25(-1 + \cos^2\theta + \cos\theta + \cos^2\theta)$
$= 25(2\cos^2\theta + \cos\theta - 1) = 25(2\cos\theta - 1)(\cos\theta + 1)$.

$dA/d\theta = 0$ for $0 < \theta < \pi/2$ when $\cos\theta = 1/2$, $\theta = \pi/3$. If $\theta = 0, \pi/3, \pi/2$ then $A = 0, 75\sqrt{3}/4, 25$ so the cross-sectional area is greatest when $\theta = \pi/3$.



40. $I = k\dfrac{\cos\phi}{\ell^2}$, k the constant of proportionality. If h is the height of the lamp above the table then $\cos\phi = h/\ell$ and $\ell = \sqrt{h^2+r^2}$ so $I = k\dfrac{h}{\ell^3} = k\dfrac{h}{(h^2+r^2)^{3/2}}$ for $h > 0$, $\dfrac{dI}{dh} = k\dfrac{r^2-2h^2}{(h^2+r^2)^{5/2}}$, $\dfrac{dI}{dh} = 0$ when $h = r/\sqrt{2}$, by the first derivative test I is maximum when $h = r/\sqrt{2}$.

41. Let L, L_1, and L_2 be as shown in the figure, then $L = L_1 + L_2 = 8\csc\theta + \sec\theta$,



$$\frac{dL}{d\theta} = -8\csc\theta\cot\theta + \sec\theta\tan\theta,\ 0 < \theta < \pi/2$$
$$= -\frac{8\cos\theta}{\sin^2\theta} + \frac{\sin\theta}{\cos^2\theta} = \frac{-8\cos^3\theta + \sin^3\theta}{\sin^2\theta\cos^2\theta};$$

$\dfrac{dL}{d\theta} = 0$ if $\sin^3\theta = 8\cos^3\theta$, $\tan^3\theta = 8$, $\tan\theta = 2$ which gives the absolute minimum for L because $\lim\limits_{\theta\to 0^+} L = \lim\limits_{\theta\to\pi/2^-} L = +\infty$.

If $\tan\theta = 2$, then $\csc\theta = \sqrt{5}/2$ and $\sec\theta = \sqrt{5}$ so $L = 8(\sqrt{5}/2) + \sqrt{5} = 5\sqrt{5}$ ft.

42. Let $x =$ number of steers per acre
$w =$ average market weight per steer
$T =$ total market weight per acre
then $T = xw$ where $w = 2000 - 50(x-20) = 3000 - 50x$
so $T = x(3000-50x) = 3000x - 50x^2$ for $0 \le x \le 60$,
$dT/dx = 3000 - 100x$ and $dT/dx = 0$ when $x = 30$. If $x = 0, 30, 60$ then $T = 0, 45000, 0$ so the total market weight per acre is largest when 30 steers per acre are allowed.

43. (a) The daily profit is

$$P = \text{(revenue)} - \text{(production cost)} = 100x - (100{,}000 + 50x + 0.0025x^2)$$
$$= -100{,}000 + 50x - 0.0025x^2$$

for $0 \le x \le 7000$, so $dP/dx = 50 - 0.005x$ and $dP/dx = 0$ when $x = 10{,}000$. Because 10,000 is not in the interval $[0, 7000]$, the maximum profit must occur at an endpoint. When $x = 0$, $P = -100{,}000$; when $x = 7000$, $P = 127{,}500$ so 7000 units should be manufactured and sold daily.

(b) Yes, because $dP/dx > 0$ when $x = 7000$ so profit is increasing at this production level.

(c) $dP/dx = 15$ when $x = 7000$, so $P(7001) - P(7000) \approx 15$, and the marginal profit is \$15.

44. (a) $R(x) = px$ but $p = 1000 - x$ so $R(x) = (1000-x)x$

(b) $P(x) = R(x) - C(x) = (1000-x)x - (3000+20x) = -3000 + 980x - x^2$

(c) $P'(x) = 980 - 2x$, $P'(x) = 0$ for $0 < x < 500$ when $x = 490$; test the points $0, 490, 500$ to find that the profit is a maximum when $x = 490$.

(d) $P(490) = 237{,}100$

(e) $p = 1000 - x = 1000 - 490 = 510$.

45. The profit is

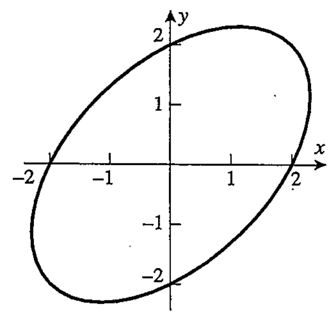
$$P = \text{(profit on nondefective)} - \text{(loss on defective)} = 100(x-y) - 20y = 100x - 120y$$

but $y = 0.01x + 0.00003x^2$ so $P = 100x - 120(0.01x + 0.00003x^2) = 98.8x - 0.0036x^2$ for $x > 0$, $dP/dx = 98.8 - 0.0072x$, $dP/dx = 0$ when $x = 98.8/0.0072 \approx 13{,}722$, $d^2P/dx^2 < 0$ so the profit is maximum at a production level of about 13,722 pounds.

53. (a) The line through (x, y) and the origin has slope y/x, and the negative reciprocal is $-x/y$. If (x, y) is a point on the ellipse, then $x^2 - xy + y^2 = 4$ and, differentiating, $2x - x(dy/dx) - y + 2y(dy/dx) = 0$.
For the desired points we have $dy/dx = -x/y$, and inserting that into the previous equation results in $2x + x^2/y - y - 2x = 0, 2xy + x^2 - y^2 - 2xy = 0, x^2 - y^2 = 0, y = \pm x$. Inserting this into the equation of the ellipse we have $x^2 \mp x^2 + x^2 = 4$ with solutions $y = x = \pm 2$ or $y = -x = \pm 2/\sqrt{3}$. Thus there are four solutions, $(2, 2), (-2, -2), (2/\sqrt{3}, -2/\sqrt{3})$ and $(-2/\sqrt{3}, 2/\sqrt{3})$.

(b) In general, the shortest/longest distance from a point to a curve is taken on the line connecting the point to the curve which is perpendicular to the tangent line at the point in question.

54. The tangent line to the ellipse at (x, y) has slope dy/dx, where $x^2 - xy + y^2 = 4$. This yields $2x - y - x dy/dx + 2y dy/dx = 0, dy/dx = (2x - y)/(x - 2y)$. The line through (x, y) that also passes through the origin has slope y/x. Check to see if the two slopes are negative reciprocals: $(2x - y)/(x - 2y) = -x/y$; $y(2x - y) = -x(x - 2y)$; $x = \pm y$, so the points lie on the line $y = x$ or on $y = -x$.



55. If $P(x_0, y_0)$ is on the curve $y = 1/x^2$, then $y_0 = 1/x_0^2$. At P the slope of the tangent line is $-2/x_0^3$ so its equation is $y - \dfrac{1}{x_0^2} = -\dfrac{2}{x_0^3}(x - x_0)$, or $y = -\dfrac{2}{x_0^3}x + \dfrac{3}{x_0^2}$. The tangent line crosses the y-axis at $\dfrac{3}{x_0^2}$, and the x-axis at $\dfrac{3}{2}x_0$. The length of the segment then is $L = \sqrt{\dfrac{9}{x_0^4} + \dfrac{9}{4}x_0^2}$ for $x_0 > 0$. For convenience, we minimize L^2 instead, so $L^2 = \dfrac{9}{x_0^4} + \dfrac{9}{4}x_0^2$, $\dfrac{dL^2}{dx_0} = -\dfrac{36}{x_0^5} + \dfrac{9}{2}x_0 = \dfrac{9(x_0^6 - 8)}{2x_0^5}$, which is 0 when $x_0^6 = 8$, $x_0 = \sqrt{2}$. $\dfrac{d^2L^2}{dx_0^2} > 0$ so L^2 and hence L is minimum when $x_0 = \sqrt{2}$, $y_0 = 1/2$.

56. If $P(x_0, y_0)$ is on the curve $y = 1 - x^2$, then $y_0 = 1 - x_0^2$. At P the slope of the tangent line is $-2x_0$ so its equation is $y - (1 - x_0^2) = -2x_0(x - x_0)$, or $y = -2x_0 x + x_0^2 + 1$. The y-intercept is $x_0^2 + 1$ and the x-intercept is $\dfrac{1}{2}(x_0 + 1/x_0)$ so the area A of the triangle is

$A = \dfrac{1}{4}(x_0^2 + 1)(x_0 + 1/x_0) = \dfrac{1}{4}(x_0^3 + 2x_0 + 1/x_0)$ for $0 \le x_0 \le 1$.

$dA/dx_0 = \dfrac{1}{4}(3x_0^2 + 2 - 1/x_0^2) = \dfrac{1}{4}(3x_0^4 + 2x_0^2 - 1)/x_0^2$ which is 0 when $x_0^2 = -1$ (reject), or when $x_0^2 = 1/3$ so $x_0 = 1/\sqrt{3}$. $d^2A/dx_0^2 = \dfrac{1}{4}(6x_0 + 2/x_0^3) > 0$ at $x_0 = 1/\sqrt{3}$ so a relative minimum and hence the absolute minimum occurs there.

57. At each point (x, y) on the curve the slope of the tangent line is $m = \dfrac{dy}{dx} = -\dfrac{2x}{(1 + x^2)^2}$ for any x, $\dfrac{dm}{dx} = \dfrac{2(3x^2 - 1)}{(1 + x^2)^3}$, $\dfrac{dm}{dx} = 0$ when $x = \pm 1/\sqrt{3}$, by the first derivative test the only relative maximum occurs at $x = -1/\sqrt{3}$, which is the absolute maximum because $\lim\limits_{x \to \pm\infty} m = 0$. The tangent line has greatest slope at the point $(-1/\sqrt{3}, 3/4)$.