EXERCISE SET 5.4

1. relative maxima at $x = 2, 6$; absolute maximum at $x = 6$; relative and absolute minima at $x = 0, 4$

2. relative maximum at $x = 3$; absolute maximum at $x = 7$; relative minima at $x = 1, 5$; absolute minima at $x = 1, 5$

3. (a) \[ y \]
   \[ x \]
   \[ 1 \]
   \[ 10 \]

(b) \[ y \]
   \[ x \]

(c) \[ y \]
   \[ x \]
   \[ 1 \]
   \[ 5 \]
   \[ 7 \]

4. (a) \[ y \]
   \[ x \]

(b) \[ y \]
   \[ x \]

(c) \[ y \]
   \[ x \]
   \[ -5 \]
   \[ 5 \]

5. $x = 1$ is a point of discontinuity of $f$.

6. Since $f$ is monotonically increasing on $(0, 1)$, one might expect a minimum at $x = 0$ and a maximum at $x = 1$. But both points are discontinuities of $f$.

7. $f'(x) = 8x - 12$, $f'(x) = 0$ when $x = 3/2$; $f(1) = 2$, $f(3/2) = 1$, $f(2) = 2$ so the maximum value is 2 at $x = 1, 2$ and the minimum value is 1 at $x = 3/2$.

8. $f'(x) = 8 - 2x$, $f'(x) = 0$ when $x = 4$; $f(0) = 0$, $f(4) = 16$, $f(6) = 12$ so the maximum value is 16 at $x = 4$ and the minimum value is 0 at $x = 0$.

9. $f'(x) = 3(x - 2)^2$, $f'(x) = 0$ when $x = 2$; $f(1) = -1$, $f(2) = 0$, $f(4) = 8$ so the minimum is $-1$ at $x = 1$ and the maximum is 8 at $x = 4$. 
10. \( f'(x) = 6x^2 + 6x - 12, f'(-1) = 0 \) when \( x = -2, 1 \); \( f(-3) = 9, f(-2) = 20, f(1) = -7, f(2) = 4 \), so the minimum is \(-7\) at \( x = 1 \) and the maximum is \( 20 \) at \( x = -2 \).

11. \( f'(x) = \frac{3}{(4x^2 + 1)^{3/2}} \), no critical points; \( f(-1) = -3/\sqrt{5}, f(1) = 3/\sqrt{5} \) so the maximum value is \( 3/\sqrt{5} \) at \( x = 1 \) and the minimum value is \(-3/\sqrt{5} \) at \( x = -1 \).

12. \( f'(x) = \frac{2(2x + 1)}{3(x^2 + x)^{1/3}} \), \( f'(x) = 0 \) when \( x = -1/2 \) and \( f'(x) \) does not exist when \( x = -1, 0 \); \( f(-2) = 2^{2/3}, f(-1) = 0, f(-1/2) = 4^{-2/3}, f(0) = 0, f(3) = 12^{2/3} \), so the maximum value is \( 12^{2/3} \) at \( x = 3 \) and the minimum value is \( 0 \) at \( x = -1, 0 \).

13. \( f'(x) = 1 - 2 \cos x, f'(x) = 0 \) when \( x = \pi/3 \); then \( f(-\pi/4) = -\pi/4 + \sqrt{2}, f(\pi/3) = \pi/3 - \sqrt{3}, f(\pi/2) = \pi/2 - 2 \), so \( f \) has a minimum of \( \pi/3 - \sqrt{3} \) at \( x = \pi/3 \) and a maximum of \(-\pi/4 + \sqrt{2} \) at \( x = -\pi/4 \).

14. \( f'(x) = \cos x + \sin x, f'(x) = 0 \) for \( x \) in \( (0, \pi) \) when \( x = 3\pi/4 \); \( f(0) = -1, f(3\pi/4) = \sqrt{2}, f(\pi) = 1 \) so the maximum value is \( \sqrt{2} \) at \( x = 3\pi/4 \) and the minimum value is \(-1 \) at \( x = 0 \).

15. \( f(x) = 1 + |9 - x^2| = \begin{cases} 10 - x^2, & |x| \leq 3 \\ -8 + x^2, & |x| > 3 \end{cases} \), \( f'(x) = \begin{cases} -2x, & |x| < 3 \\ 2x, & |x| > 3 \end{cases} \) thus \( f'(x) = 0 \) when \( x = 0 \), \( f'(x) \) does not exist for \( x \) in \((-5, 1)\) when \( x = -3 \) because \( \lim_{x \to -3^-} f'(x) \neq \lim_{x \to -3^+} f'(x) \) (see Theorem preceding Exercise 61, Section 3.3); \( f(-5) = 17, f(-3) = 1, f(0) = 10, f(1) = 9 \) so the maximum value is \( 17 \) at \( x = -5 \) and the minimum value is \( 1 \) at \( x = -3 \).

16. \( f(x) = |6 - 4x| = \begin{cases} 6 - 4x, & x \leq 3/2 \\ 6 + 4x, & x > 3/2 \end{cases} \), \( f'(x) = \begin{cases} -4, & x < 3/2 \\ 4, & x > 3/2 \end{cases} \) does not exist when \( x = 3/2 \) thus \( 3/2 \) is the only critical point in \((-3, 3)\); \( f(-3) = 18, f(3/2) = 0, f(3) = 6 \) so the maximum value is \( 18 \) at \( x = -3 \) and the minimum value is \( 0 \) at \( x = 3/2 \).

17. \( f'(x) = 2x - 1, f'(x) = 0 \) when \( x = 1/2 \); \( f(1/2) = -9/4 \) and \( \lim_{x \to \pm \infty} f(x) = +\infty \). Thus \( f \) has a minimum of \(-9/4 \) at \( x = 1/2 \) and no maximum.

18. \( f'(x) = -4(x + 1) \); critical point \( x = -1 \). Maximum value \( f(-1) = 5 \), no minimum.

19. \( f'(x) = 12x^2(1 - x) \); critical points \( x = 0, 1 \). Maximum value \( f(1) = 1 \), no minimum because \( \lim_{x \to +\infty} f(x) = -\infty \).

20. \( f'(x) = 4(x^3 + 1) \); critical point \( x = -1 \). Minimum value \( f(-1) = -3 \), no maximum.

21. No maximum or minimum because \( \lim_{x \to +\infty} f(x) = +\infty \) and \( \lim_{x \to -\infty} f(x) = -\infty \).

22. No maximum or minimum because \( \lim_{x \to +\infty} f(x) = +\infty \) and \( \lim_{x \to -\infty} f(x) = -\infty \).

23. \( \lim_{x \to -1^-} f(x) = -\infty \), so there is no absolute minimum on the interval; \( f'(x) = 0 \) at \( x \approx -2.414213562 \), for which \( y \approx -4.828427125 \). Also \( f(-5) = -13/2 \), so the absolute maximum of \( f \) on the interval is \( y \approx -4.828427125 \) taken at \( x \approx -2.414213562 \).

24. \( \lim_{x \to -1^+} f(x) = -\infty \), so there is no absolute minimum on the interval.
\( f'(x) = 3/(x+1)^2 > 0 \), so \( f \) is increasing on the interval \((-1, 5]\) and the maximum must occur at the endpoint \( x = 5 \) where \( f(5) = 1/2 \).
35. \( f'(x) = \frac{-3x^2 - 10x + 3}{x^2 + 1} \), \( f'(x) = 0 \) when \( x = \frac{1}{3} \), \( 3 \).

Then \( f(0) = 0, f \left( \frac{1}{3} \right) = 5 \ln \left( \frac{10}{9} \right) - 1 \),

\( f(3) = 5 \ln 10 - 9, f(4) = 5 \ln 17 - 12 \) and thus \( f \) has an absolute minimum of \( 5(\ln 10 - \ln 9) - 1 \) at \( x = 1/3 \) and an absolute maximum of \( 5 \ln 10 - 9 \) at \( x = 3 \).

36. \( f'(x) = (x^2 + 2x - 1)e^x \), \( f'(x) = 0 \) at \( x = -2 + \sqrt{2} \) and \( x = -1 - \sqrt{2} \) (discard), \( f(-1+\sqrt{2}) = (2-2\sqrt{2})e^{(-1+\sqrt{2})} \approx -1.25 \), absolute maximum at \( x = 2, f(2) = 3e^2 \approx 22.17 \), absolute minimum at \( x = -1 + \sqrt{2} \).

37. \( f'(x) = -\cos(x)\sin x; f'(x) = 0 \) if \( \sin x = 0 \) or if \( \cos(x) = 0 \). If \( \sin x = 0 \), then \( x = \pi \) is the critical point in \((0,2\pi)\); \( \cos(x) = 0 \) has no solutions because \(-1 \leq \cos x \leq 1 \). Thus \( f(0) = \sin(1), f(\pi) = \sin(-1) = -\sin(1) \), and \( f(2\pi) = \sin(1) \) so the maximum value is \( \sin(1) \approx 0.84147 \) and the minimum value is \(-\sin(1) \approx -0.84147 \).

38. \( f'(x) = -[\sin(x)\cos x] \cos x \); \( f'(x) = 0 \) if \( \cos x = 0 \) or if \( \sin(x) = 0 \). If \( \cos x = 0 \), then \( x = \pi/2 \) is the critical point in \((0,\pi)\); \( \sin(x) = 0 \) if \( \sin x = 0 \), which gives no critical points in \((0,\pi)\). Thus \( f(0) = 1, f(\pi/2) = \cos(1) \), and \( f(\pi) = 1 \) so the maximum value is 1 and the minimum value is \( \cos(1) \approx 0.54030 \).

39. \( f'(x) = \begin{cases} 
4, & x < 1 \\
2x - 5, & x > 1 
\end{cases} \) so \( f'(x) = 0 \) when \( x = 5/2 \), and \( f'(x) \) does not exist when \( x = 1 \) because \( \lim_{x \to 1^-} f'(x) \neq \lim_{x \to 1^+} f'(x) \) (see Theorem preceding Exercise 61, Section 3.3); \( f(1/2) = 0, f(1) = 2, f(5/2) = -1/4, f(7/2) = 3/4 \) so the maximum value is 2 and the minimum value is \(-1/4 \).

40. \( f'(x) = 2x + p \) which exists throughout the interval \((0,2)\) for all values of \( p \) so \( f'(1) = 0 \) because \( f(1) \) is an extreme value, thus \( 2 + p = 0, p = -2 \). \( f(1) = 3 \) so \( 1^2 + (-2)(1) + q = 3, q = 4 \) thus \( f(x) = x^2 - 2x + 4 \) and \( f(0) = 4, f(2) = 4 \) so \( f(1) \) is the minimum value.

41. The period of \( f(x) \) is \( 2\pi \), so check \( f(0) = 3, f(2\pi) = 3 \) and the critical points. \( f'(x) = -2\sin x - 2\sin 2x = -2\sin x(1 + 2\cos x) = 0 \) on \([0,2\pi]\) at \( x = 0, \pi, 2\pi \) and \( x = 2\pi/3, 4\pi/3 \). Check \( f(\pi) = -1, f(2\pi/3) = -3/2, f(4\pi/3) = -3/2 \).

Thus \( f \) has an absolute maximum on \((-\infty,\infty)\) of 3 at \( x = 2k\pi, k = 0, \pm 1, \pm 2, \ldots \) and an absolute minimum of \(-3/2\) at \( x = 2k\pi \pm 2\pi/3, k = 0, \pm 1, \pm 2, \ldots \).