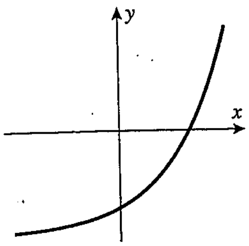
CHAPTER 5

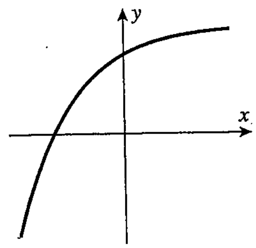
The Derivative in Graphing and Applications

EXERCISE SET 5.1

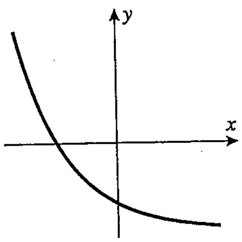
1. (a) $f' > 0$ and $f'' > 0$



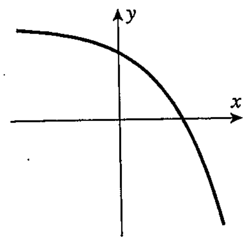
(b) $f' > 0$ and $f'' < 0$



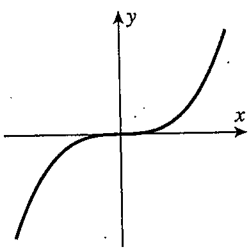
(c) $f' < 0$ and $f'' > 0$



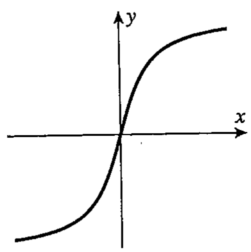
(d) $f' < 0$ and $f'' < 0$



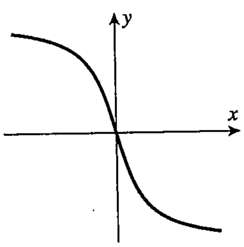
2. (a)



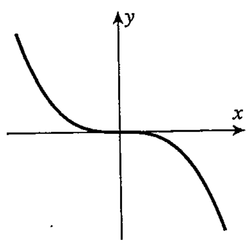
(b)



(c)



(d)



3. A: $dy/dx < 0$, $d^2y/dx^2 > 0$
B: $dy/dx > 0$, $d^2y/dx^2 < 0$
C: $dy/dx < 0$, $d^2y/dx^2 < 0$

4. A: $dy/dx < 0$, $d^2y/dx^2 < 0$
B: $dy/dx < 0$, $d^2y/dx^2 > 0$
C: $dy/dx > 0$, $d^2y/dx^2 < 0$

5. An inflection point occurs when f'' changes sign: at $x = -1, 0, 1$ and 2.

6. (a) $f(0) < f(1)$ since $f' > 0$ on $(0, 1)$.
(b) $f(1) > f(2)$ since $f' < 0$ on $(1, 2)$.
(c) $f'(0) > 0$ by inspection.
(d) $f'(1) = 0$ by inspection.
(e) $f''(0) < 0$ since f' is decreasing there.
(f) $f''(2) = 0$ since f' has a minimum there.

7. (a) $[4, 6]$
(b) $[1, 4]$ and $[6, 7]$
(c) $(1, 2)$ and $(3, 5)$
(d) $(2, 3)$ and $(5, 7)$
(e) $x = 2, 3, 5$

8.

	(1,2)	(2,3)	(3,4)	(4,5)	(5,6)	(6,7)
f'	−	−	−	+	+	−
f''	+	−	+	+	−	−

9. (a) f is increasing on $[1,3]$
(b) f is decreasing on $(-\infty,1],[3,+\infty]$
(c) f is concave up on $(-\infty,2),(4,+\infty)$
(d) f is concave down on $(2,4)$
(e) points of inflection at $x=2,4$

10. (a) f is increasing on $(-\infty,+\infty)$
(b) f is nowhere decreasing
(c) f is concave up on $(-\infty,1),(3,+\infty)$
(d) f is concave down on $(1,3)$
(e) f has points of inflection at $x=1,3$

11. $f'(x)=2(x-3/2)$
$f''(x)=2$
(a) $[3/2,+\infty)$
(b) $(-\infty,3/2]$
(c) $(-\infty,+\infty)$
(d) nowhere
(e) none

12. $f'(x)=-2(2+x)$
$f''(x)=-2$
(a) $(-\infty,-2]$
(b) $[-2,+\infty)$
(c) nowhere
(d) $(-\infty,+\infty)$
(e) none

13. $f'(x)=6(2x+1)^2$
$f''(x)=24(2x+1)$
(a) $(-\infty,+\infty)$
(b) nowhere
(c) $(-1/2,+\infty)$
(d) $(-\infty,-1/2)$
(e) $-1/2$

14. $f'(x)=3(4-x^2)$
$f''(x)=-6x$
(a) $[-2,2]$
(b) $(-\infty,-2],[2,+\infty)$
(c) $(-\infty,0)$
(d) $(0,+\infty)$
(e) 0

15. $f'(x)=12x^2(x-1)$
$f''(x)=36x(x-2/3)$
(a) $[1,+\infty)$
(b) $(-\infty,1]$
(c) $(-\infty,0),(2/3,+\infty)$
(d) $(0,2/3)$
(e) $0,2/3$

16. $f'(x)=x(4x^2-15x+18)$
$f''(x)=6(x-1)(2x-3)$
(a) $[0,+\infty)$,
(b) $(-\infty,0]$
(c) $(-\infty,1),(3/2,+\infty)$
(d) $(1,3/2)$
(e) $1,3/2$

17. $f'(x)=-\dfrac{3(x^2-3x+1)}{(x^2-x+1)^3}$
$f''(x)=\dfrac{6x(2x^2-8x+5)}{(x^2-x+1)^4}$
(a) $[\frac{3-\sqrt{5}}{2},\frac{3+\sqrt{5}}{2}]$
(b) $(-\infty,\frac{3-\sqrt{5}}{2}],[\frac{3+\sqrt{5}}{2},+\infty)$
(c) $(0,2-\frac{\sqrt{6}}{2}),(2+\frac{\sqrt{6}}{2},+\infty)$
(d) $(-\infty,0),(2-\frac{\sqrt{6}}{2},2+\frac{\sqrt{6}}{2})$
(e) $0,2-\sqrt{6}/2,2+\sqrt{6}/2$

18. $f'(x)=-\dfrac{x^2-2}{(x+2)^2}$
$f''(x)=\dfrac{2x(x^2-6)}{(x+2)^3}$
(a) $(-\infty,-\sqrt{2}),(\sqrt{2},+\infty)$
(b) $(-\sqrt{2},\sqrt{2})$
(c) $(-\infty,-\sqrt{6}),(0,\sqrt{6})$
(d) $(-\sqrt{6},0),(\sqrt{6},+\infty)$
(e) none

19. $f'(x)=\dfrac{2x+1}{3(x^2+x+1)^{2/3}}$
$f''(x)=-\dfrac{2(x+2)(x-1)}{9(x^2+x+1)^{5/3}}$
(a) $[-1/2,+\infty)$
(b) $(-\infty,-1/2]$
(c) $(-2,1)$
(d) $(-\infty,-2),(1,+\infty)$
(e) $-2,1$

20. $f'(x) = \frac{4(x-1/4)}{3x^{2/3}}$

$f''(x) = \frac{4(x+1/2)}{9x^{5/3}}$

(a) $[1/4,+\infty)$ **(b)** $(-\infty,1/4]$
(c) $(-\infty,-1/2)$, $(0,+\infty)$ **(d)** $(-1/2,0)$
(e) $-1/2, 0$

21. $f'(x) = \frac{4(x^{2/3}-1)}{3x^{1/3}}$

$f''(x) = \frac{4(x^{5/3}+x)}{9x^{7/3}}$

(a) $[-1,0],[1,+\infty)$ **(b)** $(-\infty,-1],[0,1]$
(c) $(-\infty,0)$, $(0,+\infty)$ **(d)** nowhere
(e) none

22. $f'(x) = \frac{2}{3}x^{-1/3} - 1$

$f''(x) = -\frac{2}{9x^{4/3}}$

(a) $[-1,0],[1,+\infty)$ **(b)** $(-\infty,-1],[0,1]$
(c) $(-\infty,0),(0,+\infty)$ **(d)** nowhere
(e) none

23. $f'(x) = -xe^{-x^2/2}$

$f''(x) = (-1+x^2)e^{-x^2/2}$

(a) $(-\infty,0]$ **(b)** $[0,+\infty)$
(c) $(-\infty,-1)$, $(1,+\infty)$ **(d)** $(-1,1)$
(e) $-1, 1$

24. $f'(x) = (2x^2+1)e^{x^2}$

$f''(x) = 2x(2x^2+3)e^{x^2}$

(a) $(-\infty,+\infty)$ **(b)** none
(c) $(0,+\infty)$ **(d)** $(-\infty,0)$
(e) 0

25. $f'(x) = \frac{x}{x^2+4}$

$f''(x) = -\frac{x^2-4}{(x^2+4)^2}$

(a) $[0,+\infty)$ **(b)** $(-\infty,0]$
(c) $(-2,+2)$ **(d)** $(-\infty,-2),(2,+\infty)$
(e) $-2, +2$

26. $f'(x) = x^2(1+3\ln x)$

$f''(x) = x(5+6\ln x)$

(a) $[e^{-1/3},+\infty)$ **(b)** $(0,e^{-1/3}]$
(c) $(e^{-5/6},+\infty)$ **(d)** $(0,e^{-5/6})$
(e) $e^{-5/6}$

27. $f'(x) = \frac{2x}{1+(x^2-1)^2}$

$f''(x) = -2\frac{3x^4-2x^2-2}{[1+(x^2-1)^2]^2}$

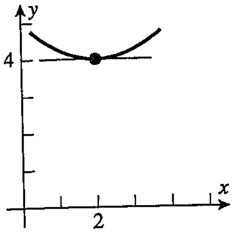
(a) $[0+\infty)$ **(b)** $(-\infty,0]$
(c) $\left(\frac{-\sqrt{1+\sqrt{7}}}{\sqrt{3}}, \frac{-\sqrt{1-\sqrt{7}}}{\sqrt{3}}\right)$, $\left(\frac{\sqrt{1-\sqrt{7}}}{\sqrt{3}}, \frac{\sqrt{1+\sqrt{7}}}{\sqrt{3}}\right)$
(d) $\left(-\infty, -\frac{\sqrt{1+\sqrt{7}}}{\sqrt{3}}\right)$, $\left(\frac{-\sqrt{1-\sqrt{7}}}{\sqrt{3}}, \frac{\sqrt{1-\sqrt{7}}}{\sqrt{3}}\right)$, $\left(\frac{\sqrt{1+\sqrt{7}}}{\sqrt{3}}, +\infty\right)$
(e) four: $\pm\frac{\sqrt{1\pm\sqrt{7}}}{3}$

28. $f'(x) = \frac{2}{3x^{1/3}\sqrt{1-x^{4/3}}}$

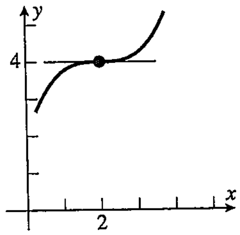
$f''(x) = \frac{2\left(-1+3x^{4/3}\right)}{9x^{4/3}\left(1-x^{4/3}\right)^{3/2}}$

(a) $[0,1]$
(b) $[-1,0]$
(c) $(-1,-3^{-3/4}),(0,3^{-3/4},1)$
(d) $(3^{-3/4},0),(3^{-3/4}$
(e) $\pm 3^{-3/4}$

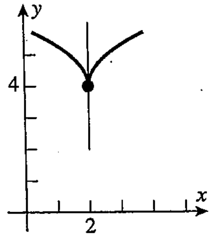
35. (a)



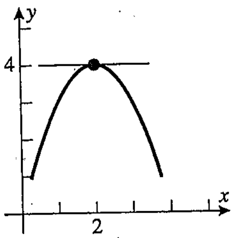
(b)



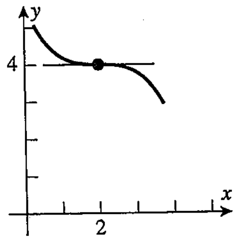
(c)



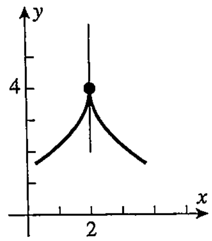
36. (a)



(b)



(c)



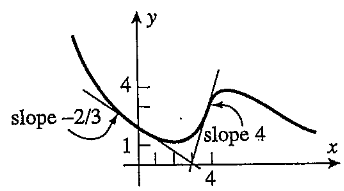
37. (a) $g(x)$ has no zeros:
There can be no zero of $g(x)$ on the interval $-\infty < x < 0$ because if there were, say $g(x_0) = 0$ where $x_0 < 0$, then $g'(x)$ would have to be positive between $x = x_0$ and $x = 0$, say $g'(x_1) > 0$ where $x_0 < x_1 < 0$. But then $g'(x)$ cannot be concave up on the interval $(x_1, 0)$, a contradiction.

There can be no zero of $g(x)$ on $0 < x < 4$ because $g(x)$ is concave up for $0 < x < 4$ and thus the graph of $g(x)$, for $0 < x < 4$, must lie above the line $y = -\frac{2}{3}x + 2$, which is the tangent line to the curve at $(0, 2)$, and above the line $y = 3(x - 4) + 3 = 3x - 9$ also for $0 < x < 4$ (see figure). The first condition says that $g(x)$ could only be zero for $x > 3$ and the second condition says that $g(x)$ could only be zero for $x < 3$, thus $g(x)$ has no zeros for $0 < x < 4$.

Finally, if $4 < x < +\infty$, $g(x)$ could only have a zero if $g'(x)$ were negative somewhere for $x > 4$, and since $g'(x)$ is decreasing there we would ultimately have $g(x) < -10$, a contradiction.

(b) one, between 0 and 4

(c) We must have $\lim_{x \to +\infty} g'(x) = 0$; if the limit were -5 then $g(x)$ would at some time cross the line $g(x) = -10$; if the limit were 5 then, since g is concave down for $x > 4$ and $g'(4) = 3$, g' must decrease for $x > 4$ and thus the limit would be ≤ 4.



38. (a) $f'(x) = 3(x - a)^2$, $f''(x) = 6(x - a)$; inflection point is $(a, 0)$

(b) $f'(x) = 4(x - a)^3$, $f''(x) = 12(x - a)^2$; no inflection points

39. For $n \geq 2$, $f''(x) = n(n - 1)(x - a)^{n-2}$; there is a sign change of f'' (point of inflection) at $(a, 0)$ if and only if n is odd. For $n = 1$, $y = x - a$, so there is no point of inflection.

40. If t is in the interval (a, b) and $t < x_0$ then, because f is increasing, $\dfrac{f(t) - f(x)}{t - x} \geq 0$. If $x_0 < t$ then $\dfrac{f(t) - f(x)}{t - x} \geq 0$. Thus $f'(x_0) = \lim_{t \to x} \dfrac{f(t) - f(x)}{t - x} \geq 0$.