EXERCISE SET 4.4

1. (a) $\displaystyle\lim_{x\to 2}\frac{x^2-4}{x^2+2x-8}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x+4)(x-2)}=\lim_{x\to 2}\frac{x+2}{x+4}=\frac{2}{3}$

(b) $\displaystyle\lim_{x\to+\infty}\frac{2x-5}{3x+7}=\frac{2-\lim\limits_{x\to+\infty}\frac{5}{x}}{3+\lim\limits_{x\to+\infty}\frac{7}{x}}=\frac{2}{3}$

2. (a) $\displaystyle\frac{\sin x}{\tan x}=\sin x\frac{\cos x}{\sin x}=\cos x$ so $\displaystyle\lim_{x\to 0}\frac{\sin x}{\tan x}=\lim_{x\to 0}\cos x=1$

(b) $\displaystyle\frac{x^2-1}{x^3-1}=\frac{(x-1)(x+1)}{(x-1)(x^2+x+1)}=\frac{x+1}{x^2+x+1}$ so $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x^3-1}=\frac{2}{3}$

3. $T_f(x)=-2(x+1)$, $T_g(x)=-3(x+1)$, limit $=2/3$

4. $T_f(x)=-\left(x-\frac{\pi}{2}\right)$, $T_g(x)=-\left(x-\frac{\pi}{2}\right)$ limit $=1$

5. $\displaystyle\lim_{x\to 0}\frac{e^x}{\cos x}=1$

6. $\displaystyle\lim_{x\to 3}\frac{1}{6x-13}=1/5$

7. $\displaystyle\lim_{\theta\to 0}\frac{\sec^2\theta}{1}=1$

8. $\displaystyle\lim_{t\to 0}\frac{te^t+e^t}{-e^t}=-1$

9. $\displaystyle\lim_{x\to\pi^+}\frac{\cos x}{1}=-1$

10. $\displaystyle\lim_{x\to 0^+}\frac{\cos x}{2x}=+\infty$

11. $\displaystyle\lim_{x\to+\infty}\frac{1/x}{1}=0$

12. $\displaystyle\lim_{x\to+\infty}\frac{3e^{3x}}{2x}=\lim_{x\to+\infty}\frac{9e^{3x}}{2}=+\infty$

13. $\lim_{x\to 0^+} \frac{-\csc^2 x}{1/x} = \lim_{x\to 0^+} \frac{-x}{\sin^2 x} = \lim_{x\to 0^+} \frac{-1}{2\sin x \cos x} = -\infty$

14. $\lim_{x\to 0^+} \frac{-1/x}{(-1/x^2)e^{1/x}} = \lim_{x\to 0^+} \frac{x}{e^{1/x}} = 0$

15. $\lim_{x\to +\infty} \frac{100x^{99}}{e^x} = \lim_{x\to +\infty} \frac{(100)(99)x^{98}}{e^x} = \cdots = \lim_{x\to +\infty} \frac{(100)(99)(98)\cdots(1)}{e^x} = 0$

16. $\lim_{x\to 0^+} \frac{\cos x/\sin x}{\sec^2 x/\tan x} = \lim_{x\to 0^+} \cos^2 x = 1$

17. $\lim_{x\to 0} \frac{2/\sqrt{1-4x^2}}{1} = 2$

18. $\lim_{x\to 0} \frac{1 - \frac{1}{1+x^2}}{3x^2} = \lim_{x\to 0} \frac{1}{3(1+x^2)} = \frac{1}{3}$

19. $\lim_{x\to +\infty} xe^{-x} = \lim_{x\to +\infty} \frac{x}{e^x} = \lim_{x\to +\infty} \frac{1}{e^x} = 0$

20. $\lim_{x\to \pi} (x-\pi)\tan(x/2) = \lim_{x\to \pi} \frac{x-\pi}{\cot(x/2)} = \lim_{x\to \pi} \frac{1}{-(1/2)\csc^2(x/2)} = -2$

21. $\lim_{x\to +\infty} x\sin(\pi/x) = \lim_{x\to +\infty} \frac{\sin(\pi/x)}{1/x} = \lim_{x\to +\infty} \frac{(-\pi/x^2)\cos(\pi/x)}{-1/x^2} = \lim_{x\to +\infty} \pi\cos(\pi/x) = \pi$

22. $\lim_{x\to 0^+} \tan x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x} = \lim_{x\to 0^+} \frac{-2\sin x\cos x}{1} = 0$

23. $\lim_{x\to (\pi/2)^-} \sec 3x \cos 5x = \lim_{x\to (\pi/2)^-} \frac{\cos 5x}{\cos 3x} = \lim_{x\to (\pi/2)^-} \frac{-5\sin 5x}{-3\sin 3x} = \frac{-5(+1)}{(-3)(-1)} = -\frac{5}{3}$

24. $\lim_{x\to \pi} (x-\pi)\cot x = \lim_{x\to \pi} \frac{x-\pi}{\tan x} = \lim_{x\to \pi} \frac{1}{\sec^2 x} = 1$

25. $y = (1-3/x)^x$, $\lim_{x\to +\infty} \ln y = \lim_{x\to +\infty} \frac{\ln(1-3/x)}{1/x} = \lim_{x\to +\infty} \frac{-3}{1-3/x} = -3$, $\lim_{x\to +\infty} y = e^{-3}$

26. $y = (1+2x)^{-3/x}$, $\lim_{x\to 0} \ln y = \lim_{x\to 0} -\frac{3\ln(1+2x)}{x} = \lim_{x\to 0} -\frac{6}{1+2x} = -6$, $\lim_{x\to 0} y = e^{-6}$

27. $y = (e^x + x)^{1/x}$, $\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(e^x+x)}{x} = \lim_{x\to 0} \frac{e^x+1}{e^x+x} = 2$, $\lim_{x\to 0} y = e^2$

28. $y = (1+a/x)^{bx}$, $\lim_{x\to +\infty} \ln y = \lim_{x\to +\infty} \frac{b\ln(1+a/x)}{1/x} = \lim_{x\to +\infty} \frac{ab}{1+a/x} = ab$, $\lim_{x\to +\infty} y = e^{ab}$

29. $y = (2-x)^{\tan(\pi x/2)}$, $\lim_{x\to 1} \ln y = \lim_{x\to 1} \frac{\ln(2-x)}{\cot(\pi x/2)} = \lim_{x\to 1} \frac{2\sin^2(\pi x/2)}{\pi(2-x)} = 2/\pi$, $\lim_{x\to 1} y = e^{2/\pi}$

30. $y = [\cos(2/x)]^{x^2}$, $\lim_{x\to +\infty} \ln y = \lim_{x\to +\infty} \frac{\ln\cos(2/x)}{1/x^2} = \lim_{x\to +\infty} \frac{(-2/x^2)(-\tan(2/x))}{-2/x^3}$

$= \lim_{x\to +\infty} \frac{-\tan(2/x)}{1/x} = \lim_{x\to +\infty} \frac{(2/x^2)\sec^2(2/x)}{-1/x^2} = -2$, $\lim_{x\to +\infty} y = e^{-2}$

31. $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x-\sin x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos x}{x\cos x + \sin x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = 0$

32. $\lim_{x\to 0} \frac{1-\cos 3x}{x^2} = \lim_{x\to 0} \frac{3\sin 3x}{2x} = \lim_{x\to 0} \frac{9}{2}\cos 3x = \frac{9}{2}$

33. $\lim_{x\to +\infty} \frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to +\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to +\infty} \frac{1}{\sqrt{1+1/x}+1} = 1/2$

34. $\lim_{x\to 0} \frac{e^x-1-x}{xe^x-x} = \lim_{x\to 0} \frac{e^x-1}{xe^x+e^x-1} = \lim_{x\to 0} \frac{e^x}{xe^x+2e^x} = 1/2$

35. $\lim_{x\to +\infty} [x-\ln(x^2+1)] = \lim_{x\to +\infty} [\ln e^x - \ln(x^2+1)] = \lim_{x\to +\infty} \ln\frac{e^x}{x^2+1}$,

$\lim_{x\to +\infty} \frac{e^x}{x^2+1} = \lim_{x\to +\infty} \frac{e^x}{2x} = \lim_{x\to +\infty} \frac{e^x}{2} = +\infty$ so $\lim_{x\to +\infty} [x-\ln(x^2+1)] = +\infty$

36. $\lim_{x\to +\infty} \ln\frac{x}{1+x} = \lim_{x\to +\infty} \ln\frac{1}{1/x+1} = \ln(1) = 0$

38. (a) $\lim_{x\to +\infty} \frac{\ln x}{x^n} = \lim_{x\to +\infty} \frac{1/x}{nx^{n-1}} = \lim_{x\to +\infty} \frac{1}{nx^n} = 0$

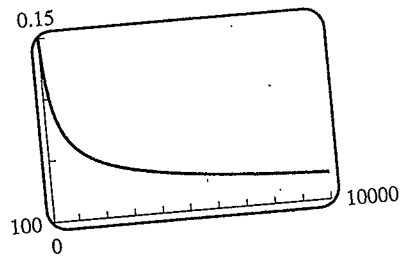
(b) $\lim_{x\to +\infty} \frac{x^n}{\ln x} = \lim_{x\to +\infty} \frac{nx^{n-1}}{1/x} = \lim_{x\to +\infty} nx^n = +\infty$

39. (a) L'Hôpital's Rule does not apply to the problem $\lim_{x\to 1} \frac{3x^2-2x+1}{3x^2-2x}$ because it is not a $\frac{0}{0}$ form.

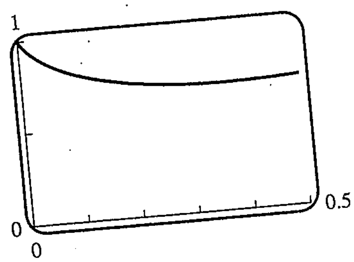
(b) $\lim_{x\to 1} \frac{3x^2-2x+1}{3x^2-2x} = 2$

40. L'Hôpital's Rule does not apply to the problem $\frac{e^{3x^2-12x+12}}{x^4-16}$, which is of the form $\frac{e^0}{0}$, and from which it follows that $\lim_{x\to 2^-}$ and $\lim_{x\to 2^+}$ exist, with values $-\infty$ if x approaches 2 from the left and $+\infty$ if from the right. The general limit $\lim_{x\to 2}$ does not exist.

41. $\lim_{x\to +\infty} \frac{1/(x\ln x)}{1/(2\sqrt{x})} = \lim_{x\to +\infty} \frac{2}{\sqrt{x}\ln x} = 0$



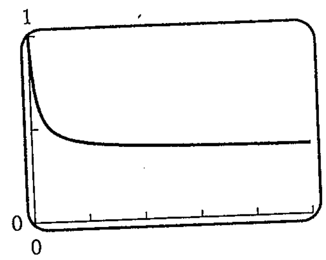
42. $y = x^x$, $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} -x = 0$, $\lim_{x\to 0^+} y = 1$



48. $y = \left(\dfrac{x+1}{x+2}\right)^x$, $\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln \dfrac{x+1}{x+2}}{1/x}$

$= \lim\limits_{x\to+\infty} \dfrac{-x^2}{(x+1)(x+2)} = -1;$

$\lim\limits_{x\to+\infty} y = e^{-1}$ is the horizontal asymptote



EXTRA CREDIT

49. (a) 0 (b) $+\infty$ (c) 0 (d) $-\infty$ (e) $+\infty$ (f) $-\infty$

50. (a) Type 0^0; $y = x^{(\ln a)/(1+\ln x)}$; $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \dfrac{(\ln a)\ln x}{1+\ln x} = \lim\limits_{x\to 0^+} \dfrac{(\ln a)/x}{1/x} = \lim\limits_{x\to 0^+} \ln a = \ln a,$
$\lim\limits_{x\to 0^+} y = e^{\ln a} = a$

(b) Type ∞^0; same calculation as Part (a) with $x \to +\infty$

(c) Type 1^∞; $y = (x+1)^{(\ln a)/x}$, $\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{(\ln a)\ln(x+1)}{x} = \lim\limits_{x\to 0} \dfrac{\ln a}{x+1} = \ln a,$
$\lim\limits_{x\to 0} y = e^{\ln a} = a$

51. $\lim\limits_{x\to+\infty} \dfrac{1+2\cos 2x}{1}$ does not exist, nor is it $\pm\infty$; $\lim\limits_{x\to+\infty} \dfrac{x+\sin 2x}{x} = \lim\limits_{x\to+\infty}\left(1+\dfrac{\sin 2x}{x}\right) = 1$

52. $\lim\limits_{x\to+\infty} \dfrac{2-\cos x}{3+\cos x}$ does not exist, nor is it $\pm\infty$; $\lim\limits_{x\to+\infty} \dfrac{2x-\sin x}{3x+\sin x} = \lim\limits_{x\to+\infty} \dfrac{2-(\sin x)/x}{3+(\sin x)/x} = \dfrac{2}{3}$

53. $\lim\limits_{x\to+\infty} (2 + x\cos 2x + \sin 2x)$ does not exist, nor is it $\pm\infty$; $\lim\limits_{x\to+\infty} \dfrac{x(2+\sin 2x)}{x+1} = \lim\limits_{x\to+\infty} \dfrac{2+\sin 2x}{1+1/x}$,
which does not exist because $\sin 2x$ oscillates between -1 and 1 as $x \to +\infty$

54. $\lim\limits_{x\to+\infty} \left(\dfrac{1}{x} + \dfrac{1}{2}\cos x + \dfrac{\sin x}{2x}\right)$ does not exist, nor is it $\pm\infty$;
$\lim\limits_{x\to+\infty} \dfrac{x(2+\sin x)}{x^2+1} = \lim\limits_{x\to+\infty} \dfrac{2+\sin x}{x+1/x} = 0$

55. $\lim\limits_{R\to 0^+} \dfrac{\frac{Vt}{L}e^{-Rt/L}}{1} = \dfrac{Vt}{L}$

56. (a) $\lim\limits_{x\to\pi/2} (\pi/2 - x)\tan x = \lim\limits_{x\to\pi/2} \dfrac{\pi/2 - x}{\cot x} = \lim\limits_{x\to\pi/2} \dfrac{-1}{-\csc^2 x} = \lim\limits_{x\to\pi/2} \sin^2 x = 1$

(b) $\lim\limits_{x\to\pi/2} \left(\dfrac{1}{\pi/2 - x} - \tan x\right) = \lim\limits_{x\to\pi/2} \left(\dfrac{1}{\pi/2 - x} - \dfrac{\sin x}{\cos x}\right) = \lim\limits_{x\to\pi/2} \dfrac{\cos x - (\pi/2 - x)\sin x}{(\pi/2 - x)\cos x}$

$= \lim\limits_{x\to\pi/2} \dfrac{-(\pi/2 - x)\cos x}{-(\pi/2 - x)\sin x - \cos x}$

$= \lim\limits_{x\to\pi/2} \dfrac{(\pi/2 - x)\sin x + \cos x}{-(\pi/2 - x)\cos x + 2\sin x} = 0$

(c) $1/(\pi/2 - 1.57) \approx 1255.765534, \tan 1.57 \approx 1255.765592;$
$1/(\pi/2 - 1.57) - \tan 1.57 \approx 0.000058$