67. (a) $\frac{d^2}{dx^2}[cf(x)] = \frac{d}{dx}\left[\frac{d}{dx}[cf(x)]\right] = \frac{d}{dx}\left[c\frac{d}{dx}[f(x)]\right] = c\frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = c\frac{d^2}{dx^2}[f(x)]$

$$\frac{d^2}{dx^2}[f(x)+g(x)] = \frac{d}{dx}\left[\frac{d}{dx}[f(x)+g(x)]\right] = \frac{d}{dx}\left[\frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]\right]$$
$$= \frac{d^2}{dx^2}[f(x)] + \frac{d^2}{dx^2}[g(x)]$$

(b) yes, by repeated application of the procedure illustrated in Part (a)

68. $\lim_{h\to 0}\frac{f'(2+h)-f'(2)}{h} = f''(2)$; $f'(x) = 8x^7 - 2$, $f''(x) = 56x^6$, so $f''(2) = 56(2^6) = 3584$.

69. (a) $f'(x) = nx^{n-1}$, $f''(x) = n(n-1)x^{n-2}$, $f'''(x) = n(n-1)(n-2)x^{n-3}, \ldots,$
$f^{(n)}(x) = n(n-1)(n-2)\cdots 1$

(b) from Part (a), $f^{(k)}(x) = k(k-1)(k-2)\cdots 1$ so $f^{(k+1)}(x) = 0$ thus $f^{(n)}(x) = 0$ if $n > k$

(c) from Parts (a) and (b), $f^{(n)}(x) = a_n n(n-1)(n-2)\cdots 1$

70. (a) If a function is differentiable at a point then it is continuous at that point, thus $f'$ is continuous on $(a, b)$ and consequently so is $f$.

(b) $f$ and all its derivatives up to $f^{(n-1)}(x)$ are continuous on $(a, b)$

71. Let $g(x) = x^n$, $f(x) = (mx+b)^n$. Use Exercise 48 in Section 3.2, but with $f$ and $g$ permuted. If $x_0 = mx_1 + b$ then Exercise 48 says that $f$ is differentiable at $x_1$ and $f'(x_1) = mg'(x_0)$. Since $g'(x_0) = nx_0^{n-1}$, the result follows.

72. $f'(x) = 2\cdot 2(2x+3) = 8x+12$

73. $f'(x) = 3\cdot 3(3x-1)^2 = 81x^2 - 54x + 9$

74. $f'(x) = 1\cdot(-1)(x-1)^{-2} = -1/(x-1)^2$

75. $f'(x) = 2\cdot 3\cdot(-2)(2x+1)^{-3} = -12/(2x+1)^3$

76. $f(x) = \frac{x+1-1}{x+1} = 1 - \frac{1}{x+1}$, and $f'(x) = -1(x+1)^{-2} = -1/(x+1)^2$

77. $f(x) = \frac{2x^2+4x+2+1}{(x+1)^2} = 2 + \frac{1}{(x+1)^2}$, so $f'(x) = -2(x+1)^{-3} = -2/(x+1)^3$

## EXERCISE SET 3.4

1. (a) $f(x) = 2x^2 + x - 1$, $f'(x) = 4x + 1$
(b) $f'(x) = (x+1)\cdot(2) + (2x-1)\cdot(1) = 4x+1$

2. (a) $f(x) = 3x^4 + 5x^2 - 2$, $f'(x) = 12x^3 + 10x$
(b) $f'(x) = (3x^2-1)\cdot(2x) + (x^2+2)\cdot(6x) = 12x^3 + 10x$

3. (a) $f(x) = x^4 - 1$, $f'(x) = 4x^3$
(b) $f'(x) = (x^2+1)\cdot(2x) + (x^2-1)\cdot(2x) = 4x^3$

4. (a) $f(x) = x^3 + 1$, $f'(x) = 3x^2$
(b) $f'(x) = (x+1)(2x-1) + (x^2-x+1)\cdot(1) = 3x^2$

5. $f'(x) = (3x^2+6)\frac{d}{dx}\left(2x-\frac{1}{4}\right) + \left(2x-\frac{1}{4}\right)\frac{d}{dx}(3x^2+6) = (3x^2+6)(2) + \left(2x-\frac{1}{4}\right)(6x)$

$= 18x^2 - \frac{3}{2}x + 12$

6. $f'(x) = (2-x-3x^3)\frac{d}{dx}(7+x^5) + (7+x^5)\frac{d}{dx}(2-x-3x^3)$

$= (2-x-3x^3)(5x^4) + (7+x^5)(-1-9x^2)$

$= -24x^7 - 6x^5 + 10x^4 - 63x^2 - 7$

7. $f'(x) = (x^3+7x^2-8)\frac{d}{dx}(2x^{-3}+x^{-4}) + (2x^{-3}+x^{-4})\frac{d}{dx}(x^3+7x^2-8)$

$= (x^3+7x^2-8)(-6x^{-4}-4x^{-5}) + (2x^{-3}+x^{-4})(3x^2+14x)$

$= -15x^{-2} - 14x^{-3} + 48x^{-4} + 32x^{-5}$

8. $f'(x) = (x^{-1}+x^{-2})\frac{d}{dx}(3x^3+27) + (3x^3+27)\frac{d}{dx}(x^{-1}+x^{-2})$

$= (x^{-1}+x^{-2})(9x^2) + (3x^3+27)(-x^{-2}-2x^{-3}) = 3 + 6x - 27x^{-2} - 54x^{-3}$

9. $f'(x) = x^2+2x+4+(2x+2)(x-2) = 3x^2$

10. $f(x) = x^4 - x^2$, $f'(x) = 4x^3 - 2x$

11. $\frac{dy}{dx} = \frac{(5x-3)\frac{d}{dx}(1) - (1)\frac{d}{dx}(5x-3)}{(5x-3)^2} = -\frac{5}{(5x-3)^2}$; $y'(1) = -5/4$

12. $\frac{dy}{dx} = \frac{(\sqrt{x}+2)\frac{d}{dx}(3) - 3\frac{d}{dx}(\sqrt{x}+2)}{(\sqrt{x}+2)^2} = -3/(2\sqrt{x}(\sqrt{x}+2)^2)$; $y'(1) = -3/18 = -1/6$

13. $\frac{dy}{dx} = \frac{(x+3)\frac{d}{dx}(2x-1) - (2x-1)\frac{d}{dx}(x+3)}{(x+3)^2}$

$= \frac{(x+3)(2) - (2x-1)(1)}{(x+3)^2} = \frac{7}{(x+3)^2}$; $\left.\frac{dy}{dx}\right|_{x=1} = \frac{7}{16}$

14. $\frac{dy}{dx} = \frac{(x^2-5)\frac{d}{dx}(4x+1) - (4x+1)\frac{d}{dx}(x^2-5)}{(x^2-5)^2}$

$= \frac{(x^2-5)(4) - (4x+1)(2x)}{(x^2-5)^2} = -\frac{4x^2+2x+20}{(x^2-5)^2}$; $\left.\frac{dy}{dx}\right|_{x=1} = \frac{13}{8}$

15. $\frac{dy}{dx} = \left(\frac{3x+2}{x}\right)\frac{d}{dx}(x^{-5}+1) + (x^{-5}+1)\frac{d}{dx}\left(\frac{3x+2}{x}\right)$

$= \left(\frac{3x+2}{x}\right)(-5x^{-6}) + (x^{-5}+1)\left[\frac{x(3)-(3x+2)(1)}{x^2}\right]$

$= \left(\frac{3x+2}{x}\right)(-5x^{-6}) + (x^{-5}+1)\left(-\frac{2}{x^2}\right)$;

$\left.\frac{dy}{dx}\right|_{x=1} = 5(-5) + 2(-2) = -29$

16. $\dfrac{dy}{dx} = (2x^7 - x^2)\dfrac{d}{dx}\left(\dfrac{x-1}{x+1}\right) + \left(\dfrac{x-1}{x+1}\right)\dfrac{d}{dx}(2x^7 - x^2)$

$$= (2x^7 - x^2)\left[\frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}\right] + \left(\frac{x-1}{x+1}\right)(14x^6 - 2x)$$

$$= (2x^7 - x^2)\cdot\frac{2}{(x+1)^2} + \left(\frac{x-1}{x+1}\right)(14x^6 - 2x);$$

$$\left.\frac{dy}{dx}\right|_{x=1} = (2-1)\frac{2}{4} + 0(14-2) = \frac{1}{2}$$

17. $f'(1) = 0$

18. $f'(1) = 1$

19. (a) $g'(x) = \sqrt{x}f'(x) + \dfrac{1}{2\sqrt{x}}f(x)$, $g'(4) = (2)(-5) + \dfrac{1}{4}(3) = -37/4$

(b) $g'(x) = \dfrac{xf'(x) - f(x)}{x^2}$, $g'(4) = \dfrac{(4)(-5) - 3}{16} = -23/16$

20. (a) $g'(x) = 6x - 5f'(x)$, $g'(3) = 6(3) - 5(4) = -2$

(b) $g'(x) = \dfrac{2f(x) - (2x+1)f'(x)}{f^2(x)}$, $g'(3) = \dfrac{2(-2) - 7(4)}{(-2)^2} = -8$

21. (a) $F'(x) = 5f'(x) + 2g'(x), F'(2) = 5(4) + 2(-5) = 10$

(b) $F'(x) = f'(x) - 3g'(x), F'(2) = 4 - 3(-5) = 19$

(c) $F'(x) = f(x)g'(x) + g(x)f'(x), F'(2) = (-1)(-5) + (1)(4) = 9$

(d) $F'(x) = [g(x)f'(x) - f(x)g'(x)]/g^2(x), F'(2) = [(1)(4) - (-1)(-5)]/(1)^2 = -1$

22. (a) $F'(x) = 6f'(x) - 5g'(x), F'(\pi) = 6(-1) - 5(2) = -16$

(b) $F'(x) = f(x) + g(x) + x(f'(x) + g'(x)),\ F'(\pi) = 10 - 3 + \pi(-1+2) = 7 + \pi$

(c) $F'(x) = 2f(x)g'(x) + 2f'(x)g(x) = 2(20) + 2(3) = 46$

(d) $F'(x) = \dfrac{(4+g(x))f'(x) - f(x)g'(x)}{(4+g(x))^2} = \dfrac{(4-3)(-1) - 10(2)}{(4-3)^2} = -21$

23. $\dfrac{dy}{dx} = \dfrac{2x(x+2) - (x^2-1)}{(x+2)^2}$,

$\dfrac{dy}{dx} = 0$ if $x^2 + 4x + 1 = 0$. By the quadratic formula,

$x = \dfrac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$. The tangent line is horizontal at $x = -2 \pm \sqrt{3}$.

24. $\dfrac{dy}{dx} = \dfrac{2x(x-1) - (x^2+1)}{(x-1)^2} = \dfrac{x^2 - 2x - 1}{(x-1)^2}$. The tangent line is horizontal when it has slope 0, i.e. $x^2 - 2x - 1 = 0$ which, by the quadratic formula, has solutions $x = \dfrac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$, the tangent line is horizontal when $x = 1 \pm \sqrt{2}$.

25. The tangent line is parallel to the line $y = x$ when it has slope 1.
$\dfrac{dy}{dx} = \dfrac{2x(x+1) - (x^2+1)}{(x+1)^2} = \dfrac{x^2 + 2x - 1}{(x+1)^2} = 1$ if $x^2 + 2x - 1 = (x+1)^2$, which reduces to $-1 = +1$, impossible. Thus the tangent line is never parallel to the line $y = x$.

**26.** The tangent line is perpendicular to the line $y = x$ when the tangent line has slope $-1$. $y = \frac{x+2+1}{x+2} = 1 + \frac{1}{x+2}$, hence $\frac{dy}{dx} = -\frac{1}{(x+2)^2} = -1$ when $(x+2)^2 = 1$, $x^2 + 4x + 3 = 0$, $(x+1)(x+3) = 0$, $x = -1, -3$. Thus the tangent line is perpendicular to the line $y = x$ at the points $(-1, 2), (-3, 0)$.

**27.** Fix $x_0$. The slope of the tangent line to the curve $y = \frac{1}{x+4}$ at the point $(x_0, 1/(x_0+4))$ is given by $\frac{dy}{dx} = \left.\frac{-1}{(x+4)^2}\right|_{x=x_0} = \frac{-1}{(x_0+4)^2}$. The tangent line to the curve at $(x_0, y_0)$ thus has the equation $y - y_0 = \frac{-1}{(x_0+4)^2}(x - x_0)$, and this line passes through the origin if its constant term $y_0 - x_0\frac{-1}{(x_0+4)^2}$ is zero. Then $\frac{1}{x_0+4} = \frac{-x_0}{(x_0+4)^2}$, $x_0 + 4 = -x_0$, $x_0 = -2$.

**28.** $y = \frac{2x+5}{x+2} = \frac{2x+4+1}{x+2} = 2 + \frac{1}{x+2}$, and hence $\frac{dy}{dx} = \frac{-1}{(x+2)^2}$, thus the tangent line at the point $(x_0, y_0)$ is given by $y - y_0 = \frac{-1}{(x_0+2)^2}(x - x_0)$ , where $y_0 = 2 + \frac{1}{x_0+2}$.
If this line is to pass through $(0, 2)$, then
$2 - y_0 = \frac{-1}{(x_0+2)^2}(-x_0)$, $\frac{-1}{x_0+2} = \frac{x_0}{(x_0+2)^2}$, $-x_0 - 2 = x_0$, $x_0 = -1$

**29.** (b) They intersect when $\frac{1}{x} = \frac{1}{2-x}$, $x = 2 - x$, $x = 1$, $y = 1$. The first curve has derivative $y = -\frac{1}{x^2}$, so the slope when $x = 1$ is m $= -1$. Second curve has derivative $y = \frac{1}{(2-x)^2}$ so the slope when $x = 1$ is m $= 1$. Since the two slopes are negative reciprocals of each other, the tangent lines are perpendicular at the point $(1, 1)$.

**30.** The curves intersect when $a/(x-1) = x^2 - 2x + 1$, or $(x-1)^3 = a$, $x = 1 + a^{1/3}$. They are perpendicular when their slopes are negative reciprocals of each other, i.e. $\frac{-a}{(x-1)^2}(2x-2) = -1$, which has the solution $x = 2a + 1$. Solve $x = 1 + a^{1/3} = 2a + 1$, $2a^{2/3} = 1$, $a = 2^{-3/2}$. Thus the curves intersect and are perpendicular at the point $(2a+1, 1/2)$ provided $a = 2^{-3/2}$.

**31.** $F'(x) = xf'(x) + f(x)$, $F''(x) = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$

**32.** (a) $F'''(x) = xf'''(x) + 3f''(x)$
(b) Assume that $F^{n)}(x) = xf^{(n)}(x) + nf^{(n-1)}(x)$ for some $n$ (for instance $n = 3$, as in part (a)). Then $F^{(n+1)}(x) = xf^{(n+1)}(x) + (1+n)f^{(n)}(x) = xf^{(n+1)}(x) + (n+1)f^{(n)}(x)$, which is an inductive proof.

**33.** $(f \cdot g \cdot h)' = [(f \cdot g) \cdot h]' = (f \cdot g)h' + h(f \cdot g)' = (f \cdot g)h' + h[fg' + f'g] = fgh' + fg'h + f'gh$

**34.** $(f_1 f_2 \cdots f_n)' = (f_1' f_2 \cdots f_n) + (f_1 f_2' \cdots f_n) + \cdots + (f_1 f_2 \cdots f_n')$

35. (a) $2(1+x^{-1})(x^{-3}+7)+(2x+1)(-x^{-2})(x^{-3}+7)+(2x+1)(1+x^{-1})(-3x^{-4})$

(b) $(x^7+2x-3)^3=(x^7+2x-3)(x^7+2x-3)(x^7+2x-3)$ so

$$\begin{aligned}\frac{d}{dx}(x^7+2x-3)^3 &= (7x^6+2)(x^7+2x-3)(x^7+2x-3)\\ &\quad +(x^7+2x-3)(7x^6+2)(x^7+2x-3)\\ &\quad +(x^7+2x-3)(x^7+2x-3)(7x^6+2)\\ &= 3(7x^6+2)(x^7+2x-3)^2\end{aligned}$$

36. (a) $-5x^{-6}(x^2+2x)(4-3x)(2x^9+1)+x^{-5}(2x+2)(4-3x)(2x^9+1)$
$+x^{-5}(x^2+2x)(-3)(2x^9+1)+x^{-5}(x^2+2x)(4-3x)(18x^8)$

(b) $(x^2+1)^{50}=(x^2+1)(x^2+1)\cdots(x^2+1)$, where $(x^2+1)$ occurs 50 times so

$$\begin{aligned}\frac{d}{dx}(x^2+1)^{50} &= [(2x)(x^2+1)\cdots(x^2+1)]+[(x^2+1)(2x)\cdots(x^2+1)]\\ &\quad +\cdots+[(x^2+1)(x^2+1)\cdots(2x)]\\ &= 2x(x^2+1)^{49}+2x(x^2+1)^{49}+\cdots+2x(x^2+1)^{49}\\ &= 100x(x^2+1)^{49} \text{ because } 2x(x^2+1)^{49} \text{ occurs 50 times.}\end{aligned}$$

37. By the product rule, $g'(x)$ is the sum of $n$ terms, each containing $n$ factors of the form $f'(x)f(x)f(x)\dots f(x)$; the function $f(x)$ occurs $n-1$ times, and $f'(x)$ occurs once. Each of these $n$ terms is equal to $f'(x)(f(x))^{n-1}$, and so $g'(x)=n(f(x))^{n-1}f'(x)$.

38. $g'(x)=10(x^2-1)^9(2x)=20x(x^2-1)^9$

39. $f(x)=\dfrac{1}{x^n}$ so $f'(x)=\dfrac{x^n\cdot(0)-1\cdot(nx^{n-1})}{x^{2n}}=-\dfrac{n}{x^{n+1}}$

40. $f(x)=g(x)h(x)$, $f'(x)=g'(x)h(x)+g(x)h'(x)$, solve for $h'$: $h'(x)=\dfrac{f'(x)-g'(x)h(x)}{g(x)}$, but $h=f/g$ so $\dfrac{d}{dx}\dfrac{f(x)}{g(x)}=\dfrac{f'(x)}{g(x)}-\dfrac{f(x)g'(x)}{g(x)^2}=\dfrac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$

## EXERCISE SET 3.5

1. $f'(x)=-4\sin x+2\cos x$

2. $f'(x)=\dfrac{-10}{x^3}+\cos x$

3. $f'(x)=4x^2\sin x-8x\cos x$

4. $f'(x)=4\sin x\cos x$

5. $f'(x)=\dfrac{\sin x(5+\sin x)-\cos x(5-\cos x)}{(5+\sin x)^2}=\dfrac{1+5(\sin x-\cos x)}{(5+\sin x)^2}$

6. $f'(x)=\dfrac{(x^2+\sin x)\cos x-\sin x(2x+\cos x)}{(x^2+\sin x)^2}=\dfrac{x^2\cos x-2x\sin x}{(x^2+\sin x)^2}$

7. $f'(x)=\sec x\tan x-\sqrt{2}\sec^2 x$

8. $f'(x)=(x^2+1)\sec x\tan x+(\sec x)(2x)=(x^2+1)\sec x\tan x+2x\sec x$