Test 3: BASIC CALCULUS I

Math 110 Fall 2007 Thursday November 29
©Prof. Ron Buckmire 7:00pm

Name: Key

Directions: Read *all* problems first before answering any of them. There are 7 pages in this test. This test is intended to be taken in 55-minutes. You may have as much time as you like (within reason!) **You may use a calculator.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

You may not discuss any answer, figure, problem, concept or question that appears on this exam with any other human individual until 10pm on Thursday November 29th.

Pledge: I, ______________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		20
2		25
3		30
4		25
BONUS		10
Total		**100**

1. L'Hôpital's Rule, The Hardest Derivative. *20 points.*
SHOW ALL YOUR WORK and **CLEARLY** indicate your final answer! If you do not remember how to simplify $f(x)^{g(x)}$, you can buy it from me for **2 points**.

(a) Evaluate $\lim_{x\to\infty}\left(\frac{1}{x^2}\right)^{\frac{1}{x}} = 1.$

$\lim_{x\to\infty}\left(\frac{1}{x^2}\right)^{\frac{1}{x}} = $ "0^0" exotic indet. form!

$= \lim_{x\to\infty} e^{\ln\left[\left(\frac{1}{x^2}\right)^{\frac{1}{x}}\right]} = \lim_{x\to\infty} e^{\frac{1}{x}\ln\left(\frac{1}{x^2}\right)} = e^{\lim_{x\to\infty}\frac{1}{x}\ln\left(\frac{1}{x^2}\right)} = e^{P} = e^{0} = 1$

$P = \lim_{x\to\infty}\frac{1}{x}\ln\left(\frac{1}{x^2}\right) = $ "$0\cdot\infty$" $= \lim_{x\to\infty}\frac{-2\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty}\frac{-\frac{2}{x}}{1} = 0$

$\ln\left(\frac{1}{x^2}\right) = -2\ln x$

(b) Evaluate $\frac{d}{dx}\left[\left(\frac{1}{x^2}\right)^{\frac{1}{x}}\right] =$

$\frac{d}{dx} e^{\frac{1}{x}\ln\left(\frac{1}{x^2}\right)} = e^{\frac{1}{x}\ln\left(\frac{1}{x^2}\right)}\cdot\left[\frac{1}{x}\ln\left(\frac{1}{x^2}\right)\right]'$ 5pts

$= \left(\frac{1}{x^2}\right)^{\frac{1}{x}}\left[-\frac{1}{x^2}\cdot\ln\left(\frac{1}{x^2}\right) + \frac{1}{x}\cdot\frac{1}{\frac{1}{x^2}}\cdot\frac{-2}{x^3}\right]$

(could ~~leave~~ stop here)

$= \left(\frac{1}{x^2}\right)^{\frac{1}{x}}\left[-\frac{1}{x^2}\ln\left(\frac{1}{x^2}\right) - \frac{2}{x^2}\right]$

5pts

2. Euler's Method, Local Linear Approximation, Concavity. *25 points.*
Suppose that you have money in an account that is earning interest at a fixed annual percentage rate (APR) of 4% compounded continuously and that you add a total of \$1000 to the account every year applied at a constant rate so that the rate of change of money M in the account is given by

$$\frac{dM}{dt} = (.04)M + 1000,$$

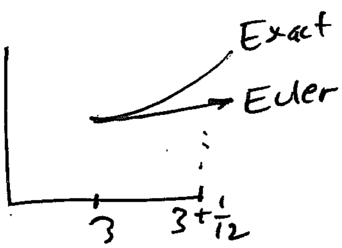
where time t is measured in years and money M is measured in dollars. Also suppose that you have \$8000 in the account at the start of year three, i.e. $M(3) = 8000$.
(a) Use an approximation to estimate how much money will be in the account at the end of January of the third year, i.e. use an approximation to estimate $M(3 + \frac{1}{12})$. $= M' \cdot \Delta t$

t	M	M'	ΔM	Δt
3	8000	$.04(8000)+1000 = 1320$	$1320 \cdot \frac{1}{12} = 110$	$1/12$
$3+1/12$	$8000 + 110 = 8110$			

$$M(3+\tfrac{1}{12}) \approx M(3) + M'(3)\cdot\frac{1}{12}$$
$$\approx 8000 + 1320 \cdot \frac{1}{12}$$
$$\approx 8000 + 110 \approx 8110$$

(b) Use the second derivative to determine if your approximation in (a) is an overestimate or an underestimate. Explain your answer.

$$\frac{d^2M}{dt^2} = \frac{d}{dt}\left(\frac{dM}{dt}\right) = \frac{d}{dt}(0.04M + 1000)$$
$$= 0.04\frac{dM}{dt} + 0$$
$$= 0.04(0.04M + 1000)$$
$$= 0.0016M + 40$$



At $M = 8000$ (or any positive M), $M'' > 0$
thus the Euler Estimate will be an UNDER-ESTIMATE of the true value, since $M(t)$ is CONCAVE UP at $t=3$, $M=8000$

3. Curve Sketching, Critical Points, Inflection Points, Extrema. *30 points.*
Consider an unknown, everywhere differentiable function $\Lambda(x)$ where the following information about $\Lambda(x)$ is given:

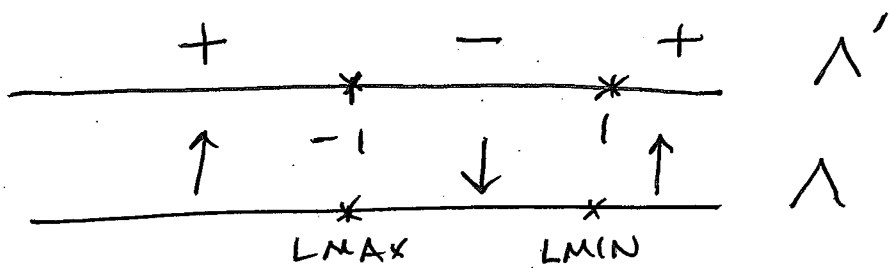
$$\lim_{x\to+\infty} \Lambda(x) = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \Lambda(x) = 0$$

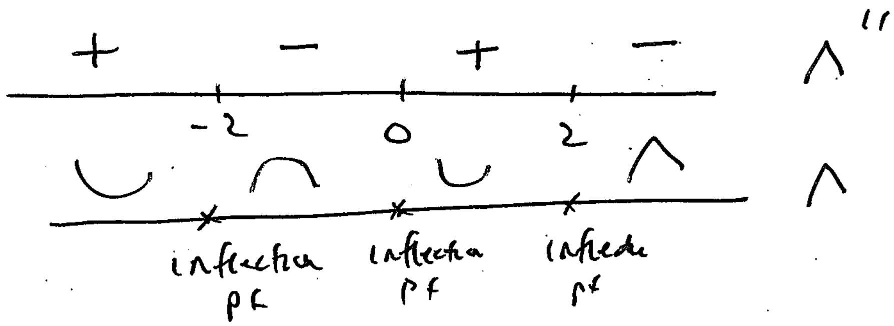
$$\Lambda(x) = \begin{cases} > 0, & \text{when } x < 0 \\ < 0, & \text{when } x > 0 \\ = 0, & \text{when } x = 0 \end{cases}$$

$$\frac{d\Lambda(x)}{dx} = \begin{cases} > 0, & \text{when } -\infty < x < -1 \text{ and } 1 < x < \infty \\ < 0, & \text{when } -1 < x < 1 \\ = 0, & \text{when } x = -1 \text{ and } x = 1 \end{cases}$$

$$\frac{d^2\Lambda(x)}{dx^2} = \begin{cases} > 0, & \text{when } -\infty < x < -2 \text{ and } 0 < x < 2 \\ < 0, & \text{when } -2 < x < 0 \text{ and } 2 < x < \infty \\ = 0, & \text{when } x = -2, x = 0 \text{ and } x = 2 \end{cases}$$

(a) (*15 points.*) Write down intervals on which the function $\Lambda(x)$ is increasing, decreasing, concave up and/or concave down. Identify any critical points and inflections points and classify any local or global extrema as either maxima or minima.





Since there's one LMAX it turns into the GMAX, and same for LMIN (is also GMIN) since $x \to \pm\infty$, $\Lambda \to 0$

The known information about $\Lambda(x)$ is repeated here for your benefit.

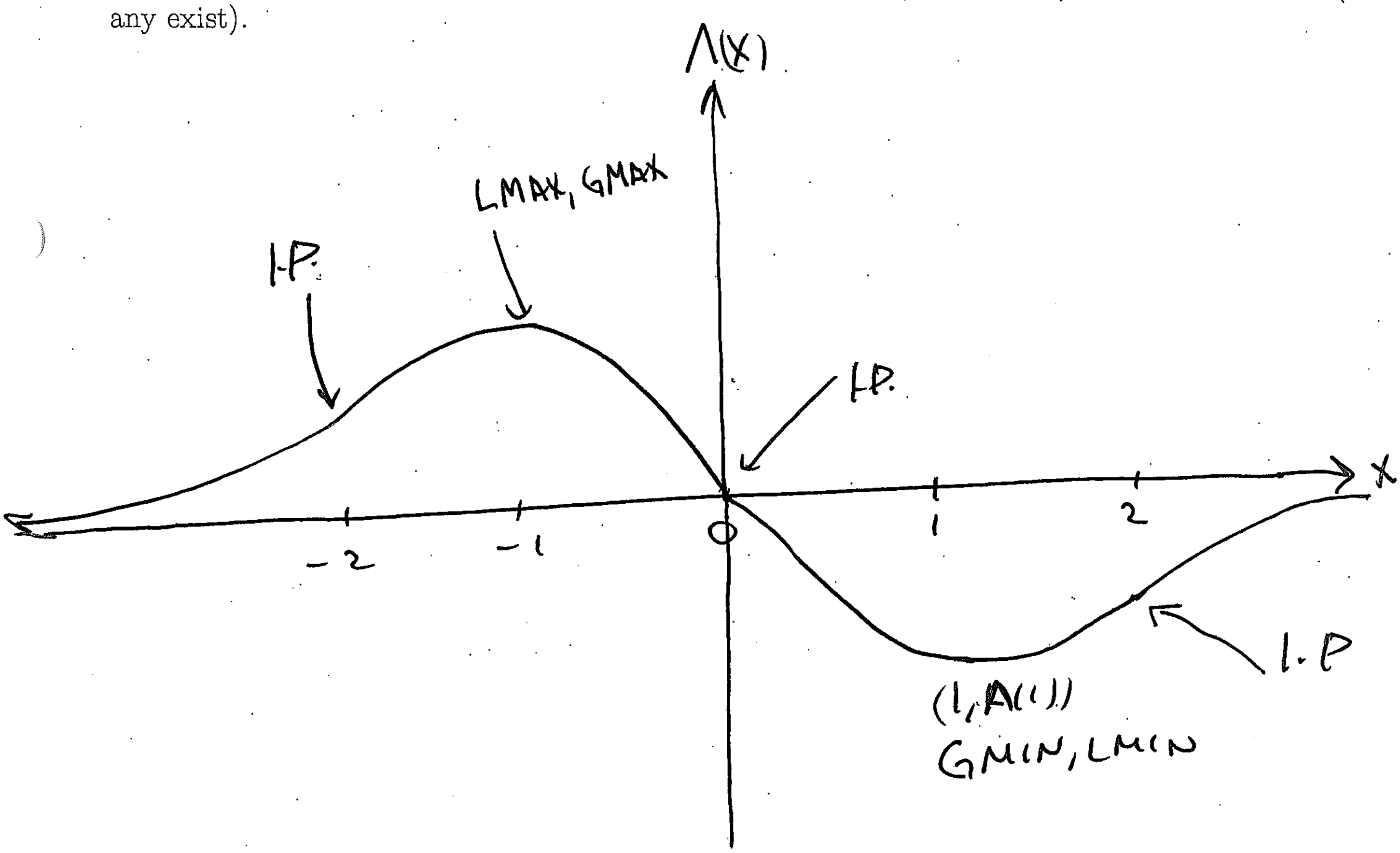
$$\lim_{x\to+\infty} \Lambda(x) = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \Lambda(x) = 0$$

$$\Lambda(x) = \begin{cases} > 0, & \text{when } x < 0 \\ < 0, & \text{when } x > 0 \\ = 0, & \text{when } x = 0 \end{cases}$$

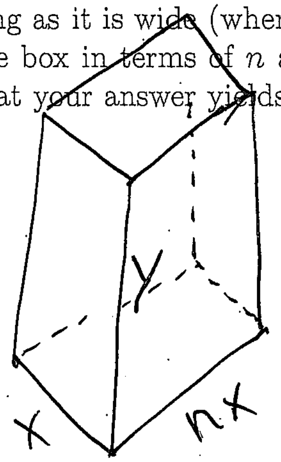
$$\frac{d\Lambda(x)}{dx} = \begin{cases} > 0, & \text{when } -\infty < x < -1 \text{ and } 1 < x < \infty \\ < 0, & \text{when } -1 < x < 1 \\ = 0, & \text{when } x = -1 \text{ and } x = 1 \end{cases}$$

$$\frac{d^2\Lambda(x)}{dx^2} = \begin{cases} > 0, & \text{when } -\infty < x < -2 \text{ and } 0 < x < 2 \\ < 0, & \text{when } -2 < x < 0 \text{ and } 2 < x < \infty \\ = 0, & \text{when } x = -2, x = 0 \text{ and } x = 2 \end{cases}$$

(b) (*15 points.*) Sketch a graph of $\Lambda(x)$ versus x in the space below. Be careful to accurately depict and NOTATE the location(s) of concavity changes (if any exist) and local extrema (if any exist).



4. Single Variable Optimization. *25 points.*
A rectangular box (with a top) is to have volume V, and its base is to be exactly n times as long as it is wide (where n is a fixed but unspecified positive number). Find the dimensions of the box in terms of n and V such that the total surface area of the box is minimized. Verify that your answer yields the global minimum.



$$V = (nx)(x)(y)$$
$$V = nx^2y$$
$$SA = 2nx^2 + 2nxy + 2xy$$
$$SA = 2nx^2 + (2n+2)xy, \quad x > 0$$
$$y = \frac{V}{nx^2} \Rightarrow SA = 2nx^2 + (2n+2)x\frac{V}{nx^2}$$
$$= 2nx^2 + 2\frac{(n+1)}{n}\frac{V}{x}$$

$$\frac{d(SA)}{dx} = 4nx - 2\frac{(n+1)}{n}\frac{V}{x^2} = 0$$
$$4nx = 2\frac{(n+1)}{n}\frac{V}{x^2}$$
$$x^3 = \frac{(n+1)V}{2n^2} \Rightarrow \frac{V}{x^3} = \frac{2n^2}{n+1}$$
$$\boxed{x = \sqrt[3]{\frac{(n+1)V}{2n^2}}} \qquad x^2 = \left[\left(\frac{n+1}{2n^2}\right)V\right]^{2/3}$$
$$y = \frac{V}{n}\cdot\frac{1}{x^2} = \frac{V}{n}\left[\left(\frac{2n^2}{n+1}\right)V\right]^{2/3}$$
$$= \frac{V^{1/3}n^{1/3}4^{1/3}}{(n+1)^{2/3}}$$
$$\boxed{y = \sqrt[3]{\frac{4Vn}{(n+1)^2}}}$$

when $x = \sqrt[3]{\frac{n+1}{2n^2}V}$,
$$\frac{d^2(SA)}{dx^2} = 4n + 4\frac{(n+1)}{n}\frac{V}{x^3}$$
$$= 4n + 4\frac{(n+1)}{n}\left(\frac{2n^2}{n+1}\right)$$
$$= 4n + 8n = 12n > 0$$

So you have a local min by 2^{nd} Deriv Test

BONUS QUESTION. *10 points.*
Consider the function $F(x) = f(x)^{g(x)}$ where $f(x) > 0$ for all x values in its domain. What are the possible results that could occur when x increases positively without bound? In other words, what are the possible values of $\lim_{x \to +\infty} f(x)^{g(x)}$? Explain your answer(s) by either providing examples of $f(x)$ and $g(x)$ that produce the different limit values you previously identified or by drawing pictures of what the graph of $F(x)$ looks like as $x \to +\infty$.

OR

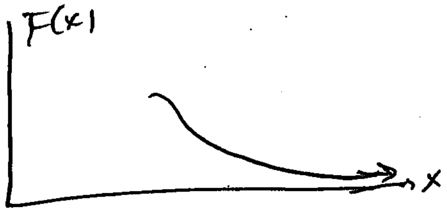
Prove that the function $F(x) = (f(x))^2$ always has a stationary point at $x = a$ when $f(x)$ has stationary points at the same location $x = a$ for all differentiable functions $f(x)$. In other words, it is always sufficient (and sometimes easier) to find the stationary points of $(f(x))^2$ instead of the stationary points of $f(x)$. Is it also true that whenever $F(x) = (f(x))^2$ possesses a stationary point at $x = a$ that $f(x)$ will also possess a stationary point at $x = a$? What are the differences between these two statements? EXPLAIN YOUR ANSWERS.

$F(x) = e^{g(x) \ln f(x)}$

If $g(x)$ is a trig fuction like $\sin(x)$ or $\cos(x)$, you can get $\lim_{x \to \infty} F(x) =$ D.N.E

$\lim_{x \to \infty} e^{\sin(x)}$

or



$\lim_{x \to \infty} e^{-x \ln x^2} = 0$

or

$\lim_{x \to \infty} e^{\frac{1}{x} \ln(\frac{1}{x^2})} = K$, $0 < K < \infty$

Statement 1 is IF f has $f'(a) = 0$ THEN $F'(a) = 0$.
Statement 2 is IF f has $F'(a) = 0$ THEN $f'(a) = 0$ [CONVERSE OF 1].

$F(x) = (f(x))^2$

$F'(x) = 2f(x)f'(x)$

$F'(a) = 2f(a)f'(a)$

If $f'(a) = 0 \Rightarrow F'(a) = 0$

But if $F'(a) = 0$ either $f(a) = 0$ OR $f'(a) = 0$,
so Statement 2 is NOT always true
while Statement 1 is.