Report on Test 3

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Grade Distribution (N=60)

Range	100+	93+	89+	84+	80+	75+	71+	65+	60+	55+	48+	41+	40-
Grade	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Frequency	8	7	8	7	4	6	4	5	2	2	1	2	4

Summary Overall class performance was the best of all the in-class exams to date. Exactly half of the class scored a B+ or higher. The mean score was 79, the median score was 84 and the mode was 87 and 91. The high score was 103. The low score was 25.

#1 L'Hôpital's Rule, The Hardest Derivative. This question was about functions of the form $f(x)^{g(x)}$ and the fact they must be re-written as $e^{g(x)\ln(f(x))}$. **(a)** $\lim_{x\to\infty}\left(\frac{1}{x^2}\right)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln\left(\frac{1}{x^2}\right)} = e^{\lim_{x\to\infty}\frac{1}{x}\ln\left(\frac{1}{x^2}\right)} = e^p = L$ where $p = \lim_{x\to\infty}\frac{1}{x}\ln\left(\frac{1}{x^2}\right) = \lim_{x\to\infty}\frac{-2\ln(x)}{x} = \lim_{x\to\infty}\frac{\frac{-2}{x}}{1} = 0 = p$ so $L = e^0 = 1 = \lim_{x\to\infty}\left(\frac{1}{x^2}\right)^{\frac{1}{x}}$.

(b) $\frac{d}{dx}\left[\left(\frac{1}{x^2}\right)^{\frac{1}{x}}\right] = \frac{d}{dx}\left[e^{\frac{1}{x}\ln\left(\frac{1}{x^2}\right)}\right] = e^{\frac{1}{x}\ln\left(\frac{1}{x^2}\right)}\frac{d}{dx}\left[\frac{1}{x}\ln\left(\frac{1}{x^2}\right)\right] = e^{\frac{1}{x}\ln\left(\frac{1}{x^2}\right)}\left[-\frac{1}{x^2}\ln\left(\frac{1}{x^2}\right) + \frac{1}{x}\frac{1}{\frac{1}{x^2}}\frac{-2}{x^3}\right]$

#2 Euler's Method, Local Linear Approximation, Concavity.
(a) Given $M' = .04M + 1000$ and $M(3) = 8000$, $M(3+\frac{1}{12}) \approx M(3) + M'(3)\frac{1}{12}$ using Euler's Method. From the differential equation, $M'(3) = .04 * M(3) + 1000 = .04*8000+1000 = 1320$. So, $M(3+\frac{1}{12}) \approx 8000 + 1320 * \frac{1}{12} = 8000 + 110 = 8110$. **(b)** $M'' = .04M' + 0 = .04*(.04M+1000) = .0016M + 40$. To determine whether our approximation in (a) is an over-estimate or under-estimate we need to know $M''(3) = .0016 * M(3) + 40 = .0016*8000+40 = 52.8 > 0$ which means that M is concave up so Euler's estimate of $M(3+\frac{1}{12})$ starting with $M(3)$ is an under-estimate.

#3 Curve Sketching, Critical Points, Inflection Points, Extrema. You can use all the given information about $\Gamma(x)$ to deduce that it has a local and global max at $x=-1$ and a local and global min at $x=1$. It also has only one root at $(0,0)$ and horizontal asymptotes as $x \to \pm\infty$ at $y=0$.

#4 Single Variable Optimization. Let the dimensions of the box be x, nx and y. We know $V = nx^2y$ and we are trying to find x and y so that the surface area A of the box is minimized. However, since a box has 6 sides, 3 pairs of which are identical, $A = 2xy + 2nx^2 + 2nxy$. However, $y = \frac{V}{nx^2}$ so that $A = 2x\frac{V}{nx^2} + 2nx^2 + 2nx\frac{V}{nx^2}$.

$$\begin{aligned}
A &= 2\frac{2V}{nx} + 2nx^2 + \frac{2V}{x} \\
\frac{d}{dx}[A] &= -\frac{2V}{nx^2} + 4nx - \frac{2V}{x^2} && \text{(Differentiate both sides with respect to } x\text{)} \\
0 &= -\frac{2V}{nx^2} + 4nx - \frac{2V}{x^2} && \text{(Set equal to zero to find critical points of A)} \\
\frac{2V}{nx^2} + \frac{2V}{x^2} &= 4nx && \text{(Get all } V \text{ on one side)} \\
\frac{2V}{x^2}\left(\frac{1}{n}+1\right) &= 4nx && \text{(Factor left hand side)} \\
2V\left(\frac{1+n}{n}\right)\frac{1}{4n} &= x^3 && \text{(Get all } x \text{ on one side)} \\
\left(\frac{1+n}{n}\right)\frac{V}{2n} &= x^3 && \text{(Simplify left hand side)} \\
\sqrt[3]{\frac{V(1+n)}{2n^2}} &= x && \text{(Take cube root of both sides)}
\end{aligned}$$

You need to check that this value of $x = x^*$ produces the minimum by using the Second Derivative Test. $A'' = \frac{4V}{nx^3} + 4n + \frac{4V}{x^3}$. We need to know the sign of $A''(x^*)$. $A''(x) = 4n + \frac{4V}{x^3}\left(\frac{1+n}{n}\right)$ and since $\frac{V}{x^3} = \frac{2n^2}{n+1}$ at the minimum value of $x = x^*$, $A''(x^*) = 4n + 4\frac{2n^2}{n+1}\frac{1+n}{n} = 4n + 8n = 12n > 0$ since $n > 0$.

The dimensions of the box with fixed volume V where one side of the base is n times the other has dimensions $\sqrt[3]{\frac{V(1+n)}{2n^2}} \times n\sqrt[3]{\frac{V(1+n)}{2n^2}} \times \frac{V}{n}\left(\frac{2n^2}{V(1+n)}\right)^{2/3}$