Test 2: Basic Calculus I

Math 110 Fall 2007 Thursday November 1
 7:00pm

Name: Key

Directions: Read *all* problems first before answering any of them. There are 7 pages in this test. This test is intended to be taken in 55-minutes. You may have as much time as you like (within reason!) **You may use a calculator.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

Pledge: I, ________________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		20
2		20
3		20
4		20
5		20
BONUS		10
Total		**100**

1. Differentiation Rules. *20 points.* Evaluate the following derivatives, making sure to clearly specify all the derivative rules you use to find your answer. Do NOT simplify your answers.

RULE 1: $(f+g)' = f' + g'$	RULE 2: $(f-g)' = f' + g'$	RULE 3: $(cf)' = cf'$
RULE 4: $(fg)' = f'g + fg'$	RULE 5: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$	RULE 6: $(f(g))' = f'(g)g'$

a. *(5 points.)* $\frac{d}{dx}[x^7 - \sqrt{9} + \frac{25}{x^{1/5}} + e^3]$

$= 7x^6 - 0 + \frac{d}{dx}(25x^{-1/5}) + 0$

$= 7x^6 - \frac{1}{5} \cdot 25 \cdot x^{-6/5}$

RULE 1, RULE 2, RULE 3

b. *(5 points.)* $\frac{d}{dr}[\sin(r)\cos(r)\tan(r)]$.

$= \frac{d}{dr}\left(\sin(r)\cos(r)\frac{\sin(r)}{\cos(r)}\right) = \frac{d}{dr}(\sin^2(r)) = 2\sin r \cos(r)$

RULE 6 OR RULE 4

c. *(5 points.)* $\frac{d}{dy}\left[\frac{4^y+5}{4y+5}\right] = \frac{\frac{d}{dy}(4^y+5)(4y+5) - (4^y+5)\frac{d}{dy}(4y+5)}{(4y+5)^2}$

RULE 5

$= \frac{(4^y \ln 4)(4y+5) - (4^y+5)4}{(4y+5)^2}$

d. *(5 points.)* $\frac{d}{ds}[e^{\ln(\sqrt{s}+1)}] = \frac{d}{ds}(\sqrt{s}+1) = \frac{1}{2\sqrt{s}}$

OR

$e^{\ln(\sqrt{s}+1)} \cdot \frac{d}{ds}\ln(\sqrt{s}+1) = e^{\ln(\sqrt{s}+1)} \cdot \frac{1}{\sqrt{s}+1} \cdot \frac{d}{ds}(\sqrt{s}+1)$

$= e^{\ln(\sqrt{s}+1)} \cdot \frac{1}{\sqrt{s}+1} \cdot \frac{1}{2\sqrt{s}}$

RULE 6

$= \sqrt{s}+1 \cdot \frac{1}{\sqrt{s}+1} \cdot \frac{1}{2\sqrt{s}} = \frac{1}{2\sqrt{s}}$

2. TRUE OR FALSE: Chain Rule, Differentials, Differentiability, Continuity, Related Rates. *20 points.*

TRUE or FALSE: put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE one way to prove this is to show there exists a counterexample and use it to prove the statement is FALSE (at least once).

(a) *5 points.* **TRUE or FALSE?** "If $f(x) = -f(-x)$ for all x, then $f'(x) = f'(-x)$ for all x."

$\boxed{\text{T}}$

$f(x) = -f(-x)$

$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} -f(-x) = -\frac{d}{dx} f(-x)$

$f'(x) = -f'(-x) \cdot -1 = -\frac{d}{d(-x)} f(-x) \frac{d(-x)}{dx}$

$f'(x) = f'(-x)$

$f'(x)$ is EVEN when $f(x)$ is ODD

(b) *5 points.* **TRUE or FALSE?** "If the length of one side of a square increases by an amount dx, the area of the square will increase by an amount twice as large, $2\,dx$."

$\boxed{\text{F}}$

$A = x^2, \quad dA = 2x\,dx$

$x = 10, \quad A = 100$

$x = 11, \quad A = 121$

$dx = 1, \quad dA = 21 \neq 2\,dx$

(c) *5 points.* **TRUE or FALSE?** "If the entire graph of a function can be drawn without picking up the drawing implement at any point then that function is differentiable everywhere."

$\boxed{\text{F}}$

CONTINUOUS FUNCTION AT 0, BUT $F'(0)$ DOES NOT EXIST

CONTINUITY DOES NOT IMPLY DIFFERENTIABILITY!

Differentiability $\Longrightarrow$ Continuity

(d) *5 points.* **TRUE or FALSE?** "If at $t = 3$ $\frac{dy}{dt} = 4$ snarfs per minute and $\frac{dx}{dt} = -2$ Muggles per minute, then $\frac{dy}{dx} = -\frac{1}{2}$ Muggles per snarf at $t = 3$."

$\boxed{\text{F}}$

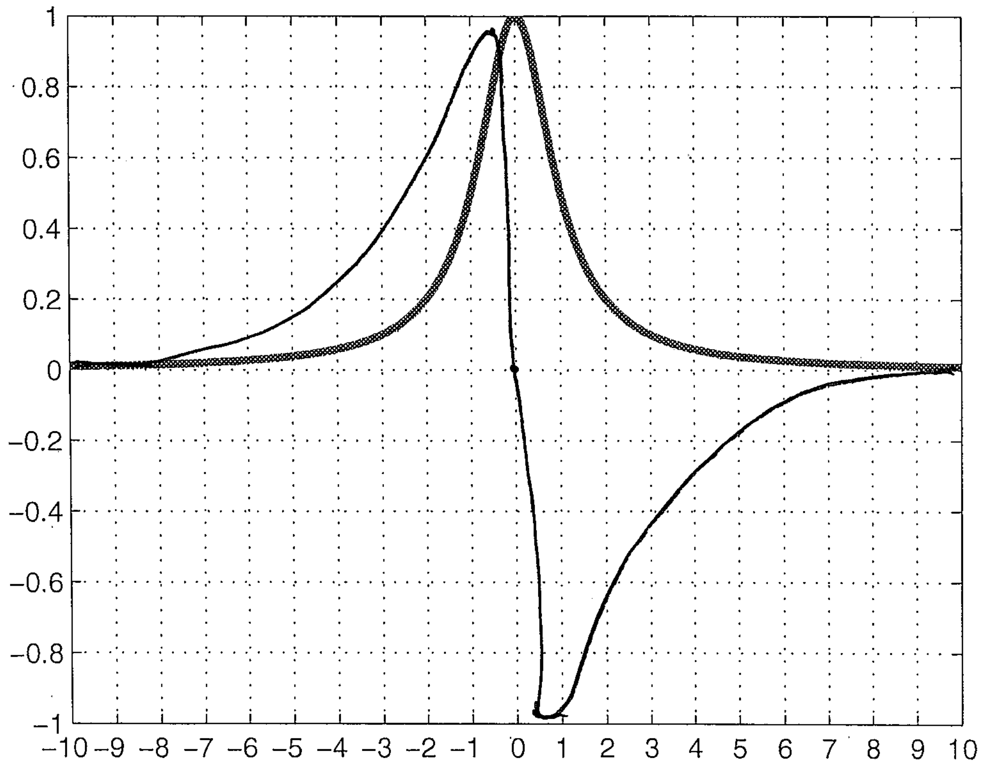
$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$4 \frac{\text{snarfs}}{\text{min}} = -\frac{1}{2} \frac{\text{Muggles}}{\text{snarf}} \cdot -2 \frac{\text{Muggles}}{\text{min}}$

$\frac{dy}{dx}$ must equal $-2 \frac{\text{snarf}}{\text{Muggle}}$ NOT $-\frac{1}{2} \frac{\text{Muggle}}{\text{snarf}}$

3. **Visualization of Derivative Function.** *20 points.* Consider the graph of the following function $f(x)$. This problem is about interpreting the derivative function graphically and conceptually.

(a) (*10 points.*) On the same axes as $f(x)$ below, sketch a graph of $f'(x)$.



(b) (*10 points.*) Write down a paragraph explaining all of the particular notable features of the graph of $f'(x)$ you drew, providing the reasons for including these features in your graph. For example, notable features of the given graph $f(x)$ are that it has a single horizontal asymptote as $x \to \infty$ or $x \to -\infty$ and possesses a single maximum value at $x = 0$ and is always positive. You should describe and explain the notable features of your sketch of the graph of $f'(x)$ similarly.

When $x \to \infty$, $f(x) \to 0$ and when $x \to -\infty$, $f(x) \to 0$,
This means $f(x)$ approaches a constant so $f' \to 0$
Somewhere near the peak at $(0,1)$ f' stops increasing
and decreases to a value of 0 at $x=0$, since f has a
peak there. f switches from increasing at 0^- to
decreasing at 0^+ very rapidly so f' switches from
large positive through $(0,0)$ to large negative.
f' has a local max value at 0^- and a local min value at
0^+. For $x>0$, f' must be negative, since $f \downarrow$ for $x>0$
Note f is an even function so f' is an odd function

4. Definition of Derivative, Limits. *20 points.*

Consider the following limit

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

(a) (*10 points.*) Interpret the above limit as the definition of the derivative of some function $f(x)$ evaluated at some point a. Determine $f(x)$, a and confirm that the value of the limit above is indeed equal to $f'(a)$ for your choice of $f(x)$ and a.

$$\lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{h\to 0} \frac{e^h - e^0}{h - 0} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

where $f(x) = e^x$ and $a = 0$ $\quad f(a) = e^0 = 1$

$$f'(a) = e^x\Big|_{x=0} = e^0 = 1$$

(b) (*10 points.*) Use the limit definition of the derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

and the value of the limit at the top of the page to prove that $f'(x) = f(x)$ when $f(x) = e^x$. Be careful to clearly state what limit rules you are applying during each step of your proof that $f'(x) = f(x)$ when $f(x) = e^x$.

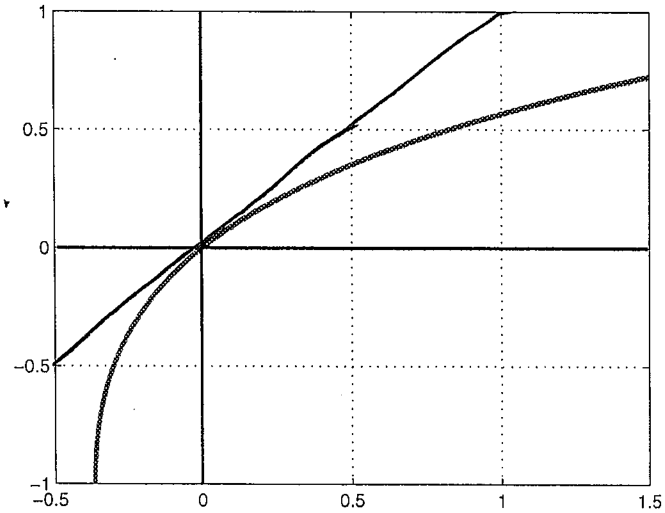
$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$\overset{\text{Algebra}}{=} \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1$$

$$f'(x) = e^x = f(x)$$

5. Chain Rule, Tangent Lines, Implicit Differentiation, Logarithmic Differentiation.
20 points.

Consider the graph of the implicit curve $x = ye^y$ given below



(a) (*10 points.*) Show that $\dfrac{dy}{dx} = \dfrac{y}{x}\dfrac{1}{y+1}$.

$$\frac{dx}{dy} = y\frac{d}{dy}(e^y) + \frac{dy}{dy}e^y = ye^y + e^y$$

$$\frac{dy}{dx} = \frac{1}{ye^y+e^y} = \frac{1}{e^y(y+1)} = \frac{1}{e^y}\cdot\frac{1}{y+1} = \frac{1}{\frac{x}{y}}\frac{1}{y+1}$$

$$= \frac{y}{x}\frac{1}{y+1}$$

(b) (*10 points.*) Find the equation of the tangent line to the implicitly defined curve $x = ye^y$ at the point $(0,0)$. If the tangent line does not exist at that point, explain why.

At $(0,0)$, $\dfrac{dy}{dx} = \dfrac{1}{0\cdot e^0 + e^0} = \dfrac{1}{1}$

$y = f(x)$
even though
f is unknown

$f'(0) = \left.\dfrac{dy}{dx}\right|_{\substack{x=0\\y=0}}$

$f(0) = 0$

$y - f(0) = f'(0)(x-0)$

$y - 0 = 1(x-0)$

$y = x$ is
the equation of the tangent line to $x = ye^y$ at $(0,0)$

BONUS QUESTION. *10 points.* Although the function $y = f(x)$ implicitly defined through the equation $x = ye^y$ can not be found explicitly, you found its derivative in Question 4. For what values of y will the inverse function $x = f^{-1}(y)$ exist? For what values of x is the function $y = f(x)$ invertible? If possible, find an explicit formula for the inverse of the implicitly defined function $y = f(x)$ from Question 4? EXPLAIN YOUR ANSWER. Regardless of whether you can find $f^{-1}(y)$ explicitly, can you obtain its derivative explicitly? If so, write down a formula for it.

OR

Use a local linear approximation to the implicitly defined function $x = ye^y$ near $(0, 0)$ to approximate the solution to the equation $0.5 = ye^y$. Draw a picture to indicate whether your approximation is greater than or less than the actual exact answer and discuss how you would improve your estimated value.

$f^{-1}(y)$ will exist for the values of the range that $f(x)$ is increasing or f is 1 to 1.

$$\frac{dy}{dx} = \frac{1}{e^y(y+1)} > 0 \text{ when } y+1 > 0 \text{ or } y > -1$$

$x = ye^y$ Since $y = -1 \Rightarrow x = -1e^{-1} = -\frac{1}{e}$

So $f(x)$ is invertible for $x > -\frac{1}{e}$

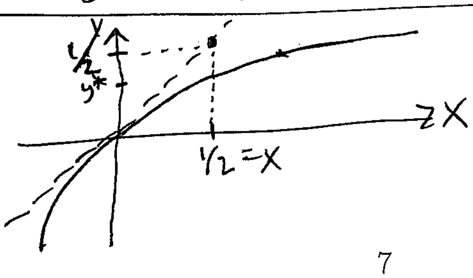
The inverse $x = f^{-1}(y) = ye^y$, $y > -1$

$y = f(x)$ is unknown on $x > -\frac{1}{e}$

$$\frac{dx}{dy} = \frac{d}{dy}(f^{-1}(y)) = \frac{d}{dy}(ye^y) = ye^y + 1 \cdot e^y = ye^y + e^y$$

$$\frac{dy}{dx} = \frac{1}{ye^y + e^y} = \frac{1}{e^y(y+1)}$$

OR



Improve your estimate by finding tangent line at value closer to $x = 1/2$

Sol'n to $\frac{1}{2} = ye^y$ is some value y^* on the curve $x = ye^y$

Use tangent line $y = x$ to approximate it, so $y = 1/2$ is estimate, which is above exact y^*