Test 2: Basic Calculus I

Math 110 Fall 2007
©Prof. Ron Buckmire

Name: Key

Directions: Read all problems first before answering any of them. There are 7 pages in this test. This test is intended to be taken in 55-minutes. You may have as much time as you like (within reason!) You may use a calculator. You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

Pledge: I, ____________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

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1. Differentiation Rules. 20 points. Evaluate the following derivatives, making sure to clearly specify all the derivative rules you use to find your answer. Do NOT simplify your answers.

RULE 1: \((f + g)' = f' + g'\)  
RULE 2: \((f - g)' = f' - g'\)  
RULE 3: \((cf)' = cf'\)

RULE 4: \((fg)' = f'g + fg'\)  
RULE 5: \(\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}\)  
RULE 6: \((f(g))' = f'(g)g'\)

a. (5 points.) \(\frac{d}{dx}[x^7 - \sqrt{x} + \frac{25}{x^{1/5}} + e^3]\)

\[
= 7x^6 - 0 + \frac{d}{dx}(25x^{-1/5}) + 0
= 7x^6 - \frac{1}{5} \cdot 25 \cdot x^{-6/5}
\text{RULE 1, RULE 2, RULE 3}
\]

b. (5 points.) \(\frac{d}{dr}[(\sin(r) \cos(r)) \tan(r)]\)

\[
= \frac{d}{dr}(\sin(r) \cos(r) \frac{\sin(r)}{\cos(r)}) = \frac{d}{dr}(\sin^2(r)) = 2 \sin(r) \cos(r)
\text{RULE 6, RULE 4}
\]

c. (5 points.) \(\frac{d}{dy}[\frac{4y + 5}{4y + 5}]\)

\[
= \frac{d}{dy}(4y + 5)(4y + 5) - (4y + 5) \frac{d}{dy}(4y + 5)
= \frac{(4y + 5)^2 - (4^2 + 5)}{(4y + 5)^2}
\]

\[
= \frac{(4y + 5)(4y + 5) - (4^2 + 5)}{(4y + 5)^2}
\]

\[
= \frac{(4y + 5)(4y + 5) - (4^2 + 5)}{(4y + 5)^2}
\]

d. (5 points.) \(\frac{d}{ds}[\ln(\sqrt{s} + 1)]\)

\[
= \frac{d}{ds}(\sqrt{s} + 1) = \frac{1}{2\sqrt{s}}
\]

\[
= \frac{\ln(\sqrt{s} + 1)}{\sqrt{s} + 1} \cdot \frac{1}{\sqrt{s} + 1} \cdot \frac{1}{\sqrt{s} + 1}
= \frac{1}{2\sqrt{s}}
\]

RULE 6

\[
= \sqrt{s + 1} \cdot \frac{1}{\sqrt{s} + 1} \cdot \frac{1}{\sqrt{s} + 1} = \frac{1}{2\sqrt{s}}
\]
2. TRUE OR FALSE: Chain Rule, Differentials, Differentiability, Continuity, Related Rates. 20 points.

TRUE or FALSE: put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE one way to prove this is to show there exists a counterexample and use it to prove the statement is FALSE (at least once).

(a) 5 points. TRUE or FALSE? “If \( f(x) = -f(-x) \) for all \( x \), then \( f'(x) = f'(-x) \) for all \( x \).”

\[
\begin{align*}
\text{True} & \quad f(x) = -f(-x) \\
\text{Differentiate wrt } x & \quad f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} f(-x) = -f'(-x) \\
\text{Substitute } f(x) & \quad f'(x) = -f'(-x) \cdot (-1) = f'(-x) \\
\text{Therefore } f'(x) & \quad \text{is even when } f(x) \text{ is odd.}
\end{align*}
\]

(b) 5 points. TRUE or FALSE? “If the length of one side of a square increases by an amount \( dx \), the area of the square will increase by an amount twice as large, \( 2 \) \( dx \).”

\[
\begin{align*}
A &= x^2, \quad dA = 2x \, dx \\
x &= 10, \quad A = 100 \\
x &= 11, \quad A = 121 \\
dx = 1, \quad dA = 21 \neq 2 \, dx
\end{align*}
\]

(c) 5 points. TRUE or FALSE? “If the entire graph of a function can be drawn without picking up the drawing implement at any point then that function is differentiable everywhere.”

\[
\begin{align*}
\text{Function is continuous at } 0. \quad \text{But } f'(0) & \quad \text{not defined.}
\end{align*}
\]

(d) 5 points. TRUE or FALSE? “If at \( t = 3 \) \( \frac{dy}{dt} = 4 \) snarfs per minute and \( \frac{dx}{dt} = -2 \) Muggles per minute, then \( \frac{dy}{dx} = \frac{1}{2} \) Muggles per snarf at \( t = 3 \).”

\[
\begin{align*}
\frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} \\
\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} \\
\frac{4 \text{ snarfs}}{\text{min}} &= \frac{1}{2} \frac{\text{Muggles}}{\text{snarf}} \\
-2 \text{ Muggle} &= \frac{1}{2} \text{ Muggle per snarf}
\end{align*}
\]
3. **Visualization of Derivative Function.** 20 points. Consider the graph of the following function $f(x)$. This problem is about interpreting the derivative function graphically and conceptually.

(a) (10 points.) On the same axes as $f(x)$ below, sketch a graph of $f'(x)$.

![Graph of $f(x)$ with axes labeled from -10 to 10 on the x-axis and from -1 to 1 on the y-axis.]

(b) (10 points.) Write down a paragraph explaining all of the particular notable features of the graph of $f'(x)$ you drew, providing the reasons for including these features in your graph. For example, notable features of the given graph $f(x)$ are that it has a single horizontal asymptote as $x \to \infty$ or $x \to -\infty$ and possesses a single maximum value at $x = 0$ and is always positive. You should describe and explain the notable features of your sketch of the graph of $f'(x)$ similarly.

When $x \to \infty$, $f(x) \to 0$ and when $x \to -\infty$, $f(x) \to 0$,

This means $f(x)$ approaches a constant so $f' \to 0$ somewhere near the peak at $0$. If $f'$ stops increasing and decreases to a value of $0$ at $x=0$, since $f$ has a peak there, $f$ switches from increasing at $0^-$ to decreasing at $0^+$ very rapidly so $f'$ switches from large positive through $0^+$ to large negative.

$f'$ has a local max value at $0^-$ and a local min value at $0^+$. For $x > 0$, $f'$ must be negative, since $f \downarrow$ for $x > 0$.

Note $f$ is an even function so $f'$ is an odd function.
4. Definition of Derivative, Limits. 20 points.

Consider the following limit

\[
\lim_{h \to 0} \frac{e^h - 1}{h} = 1.
\]

(a) (10 points.) Interpret the above limit as the definition of the derivative of some function \(f(x)\) evaluated at some point \(a\). Determine \(f(x)\) and \(a\) and confirm that the value of the limit above is indeed equal to \(f'(a)\) for your choice of \(f(x)\) and \(a\).

\[
\lim_{h \to 0} \frac{e^h - 1}{h} = \lim_{h \to 0} \frac{e^h - e^0}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)
\]

where \(f(x) = e^x\) and \(a = 0\) \(f(a) = e^0 = 1\)

\[
f'(a) = e^x \bigg|_{x=0} = e^0 = 1
\]

(b) (10 points.) Use the limit definition of the derivative

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

and the value of the limit at the top of the page to prove that \(f'(x) = f(x)\) when \(f(x) = e^x\). Be careful to clearly state what limit rules you are applying during each step of your proof that \(f'(x) = f(x)\) when \(f(x) = e^x\).

\[
f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}
\]

Algebra
\[
= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \cdot 1
\]

\[
f'(x) = e^x = f(x)
\]
20 points.

Consider the graph of the implicit curve \( x = ye^y \) given below

(a) \(10\) points.\) Show that \( \frac{dy}{dx} = \frac{y}{x y + 1} \).

\[
\frac{dx}{dy} = y e^y + e^y
\]

\[
\frac{dy}{dx} = \frac{1}{y e^y + e^y} = \frac{1}{e^y (y+1)} = \frac{1}{e^y} \cdot \frac{1}{y+1} = \frac{1}{y} \cdot \frac{1}{y+1}
\]

\[
= \frac{y}{x y + 1}
\]

(b) \(10\) points.\) Find the equation of the tangent line to the implicitly defined curve \( x = ye^y \) at the point \((0, 0)\). If the tangent line does not exist at that point, explain why.

At \( e^y = 1 \), \( \frac{dy}{dx} = \frac{1}{0 \cdot e^y + e^y} = \frac{1}{1} \)

\[
Y - f(0) = f'(0) (x - 0)
\]

\[
Y - 0 = 1 (x - 0)
\]

\[ Y = x \]

is the equation of the tangent line to \( x = ye^y \) at \((0, 0)\)
**BONUS QUESTION. 10 points.** Although the function \( y = f(x) \) implicitly defined through the equation \( x = ye^y \) cannot be found explicitly, you found its derivative in Question 4. For what values of \( y \) will the inverse function \( x = f^{-1}(y) \) exist? For what values of \( x \) is the function \( y = f(x) \) invertible? If possible, find an explicit formula for the inverse of the implicitly defined function \( y = f(x) \) from Question 4? EXPLAIN YOUR ANSWER. Regardless of whether you can find \( f^{-1}(y) \) explicitly, can you obtain its derivative explicitly? If so, write down a formula for it.

OR

Use a local linear approximation to the implicitly defined function \( x = ye^y \) near \((0,0)\) to approximate the solution to the equation \( 0.5 = ye^y \). Draw a picture to indicate whether your approximation is greater than or less than the actual exact answer and discuss how you would improve your estimated value.

\[
\frac{dy}{dx} = \frac{1}{y e^y + e^y} > 0 \quad \text{when} \quad y + 1 > 0 \quad \text{or} \quad y > -1
\]

\( x = ye^y \) \quad \text{Since} \quad y = -1 \quad \Rightarrow \quad x = -1e^{-1} = -\frac{1}{e}

So \( f(x) \) is invertible for \( x > -\frac{1}{e} \)

The inverse \( x = f^{-1}(y) = ye^y, \ y > -1 \)

\( y = f(x) \) is unknown on \( x > -\frac{1}{e} \)

\[
\frac{dx}{dy} = \frac{d}{dy} (f^{-1}(y)) = \frac{d}{dy} (ye^y) = ye^y + e^y = ye^y + e^y
\]

\[
\frac{dy}{dx} = \frac{1}{ye^y + e^y} = \frac{1}{e^y(y + 1)}
\]

\[
\text{Solve} \quad \frac{1}{2} = ye^y \quad \text{is some value} \quad y^* \text{ on the curve } x = ye^y
\]

Use tangent line \( y = x \) to approximate it, so \( y = \frac{1}{2} \) is estimate, which is above upper