Math 110 Fall 2007 — Basic Calculus I

Report on Test 2 — Prof. Ron Buckmire

Grade Distribution (N=61)

Range	99+	90+	85+	80+	75+	70+	65+	60+	55+	50+	45+	40+	39-
Grade	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Frequency	1	4	5	8	8	7	5	5	3	6	5	2	2

Summary Overall class performance was about the same as on the first exam; pretty good. There was less variation in the scores on Exam 2 than on Exam 1! The mean score was 69 with a standard deviation of 16, the median score was 73 and the mode was 74. The high score was a 108. The low score was 30.

#1 Differentiation Rules. This question was to test basic differentiation rules and technique.

(a) $\frac{d}{dx}[x^7 - \sqrt{9} + \frac{25}{x^{1/5}} + e^3] = 7x^6 - 0 + 25(-\frac{1}{5}x^{-4/5}) + 0$ Using Rule 1, 2 and 3 and Power Rule. $\sqrt{9}$ and e^3 are CONSTANTS so have zero derivatives.

(b) $\frac{d}{dr}[\sin(r)\cos(r)\tan(r) = 2\sin(r)\cos(r)$ using either Rule 4 or Rule 6. Simplify first, because $\sin(r)\cos(r)\tan(r) = \sin(r)\sin(r)$. If you do use Product Rule (Rule 4) directly, the derivative is $(\sin(r))'\cos(r)\tan(r)+\sin(r)(\cos(r))'\tan(r)+\sin(r)\cos(r)(\tan(r))' = \cos(r)\cos(r)\tan(r) + \cos(r)(-\sin(r))\tan(r) + \sin(r)\cos(r)\sec^2(r)$. Do not simplify!!

(c) $\frac{d}{dy}\left[\frac{4^y+5}{4y+5}\right] = \frac{(4^y+5)'(4y+5) - (4^y+5)(4y+5)'}{(4y+5)^2} = \frac{(4^y\ln(4))(4y+5) - (4^y+5)(4)}{(4y+5)^2}$. Using the Quotient Rule (Rule 5).

(d) $\frac{d}{ds}[e^{\ln(\sqrt{s}+1)}] = e^{\ln(\sqrt{s}+1)}(\ln(\sqrt{s}+1))' = e^{\ln(\sqrt{s}+1)}(\frac{1}{\sqrt{s}+1}(\sqrt{s}+1))' = e^{\ln(\sqrt{s}+1)}(\frac{1}{\sqrt{s}+1}\frac{1}{2\sqrt{s}}$

You can also do this problem by recognizing that $[e^{\ln(\sqrt{s}+1)}] = \sqrt{s}+1$ so its derivtiave is simply $\frac{1}{2\sqrt{s}}$ using the Power Rule.

#2 Chain Rule, Differentials, Differentiability, Continuity, Related Rates. TRUE OR FALSE Question.

(a) "If $f(x) = -f(-x)$ for all x, then $f'(x) = f'(-x)$ for all x." **TRUE.** This statement is saying the derivative of all ODD functions is an EVEN function. Although you may know this intuitively, you need to PROVE it is true for all functions $f(x)$ using the definition. You can do this by differentiating the given equation. $\frac{d}{dx}[f(x)] = \frac{d}{dx}[-f(-x)] = -\frac{d}{dx}[f(-x)] = -f'(-x)\cdot\frac{d}{dx}[-x] = -f'(-x)\cdot -1 = f'(-x)$. Thus $f'(x) = f(-x)$. **(b)** "If the length of one side of a square increases by an amount dx, the area of the square will increase by an amount twice as large, 2 dx." **FALSE.** It is true that the area A of a square of side x is $A = x^2$ and $dA = 2xdx$ and $\frac{dA}{A} = 2\frac{dx}{x}$ but this is NOT what the question is asking. It posits a very specific scenario which is clearly false by choosing almost any size square and increasing a side by an amount and then recomputing the area. **(c)** "If the entire graph of a function can be drawn without picking up the drawing implement at any point then that function is differentiable everywhere." This about converting English into mathematics. The statement is equivalent to saying "Continuity implies differentiability." If there's ONE thing you remember from Calculus 1, let it be that this statement is **FALSE**. A counterexample is the function $f)x) = |x|$ which is everywhere continuous but not diferentiable at $x = 0$. The theorem to remember (forever!) is that "Differentiability implies continuity" AND its contrapositive "NOT continuous implies NOT differentiable." **(d)** "If at $t = 3$ $\frac{dy}{dt} = 4$ snarfs per minute and $\frac{dx}{dt} = -2$ Muggles per minute, then $\frac{dy}{dx} = -\frac{1}{2}$ Muggles per snarf at $t = 3$." Using Related rates $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. This means that 4 snarfs/minute=$\frac{dy}{dx}$(-2 Muggles per minute), so that $\frac{dy}{dx}$ must have UNITS of snarfs/Muggle but also be equal to -2. This is the reciprocal of the value given, so the statement is **FALSE.**

#3 Visualization of Derivatives. This question is for both visual learning styles and verbal learning styles. Your sketch of the derivative function must have these features: go through the origin $(0,0)$ (because $f(x)$ has a maximum at $x = 0$), go to zero asymptotically as $x \to \pm\infty$ (because $f(x)$ becomes a constant function as $x \to \pm\infty$), have $f'(x) > 0$ when $x < 0$ because that's when f is increasing and have $f'(x) < 0$ when $x > 0$ because that's where f is decreasing. $f'(x)$ should have a local maximum (near 0-) and local minimum value (near 0+) since the rate of increase of the function changes as it approaches the maximum at $x = 0$ and again the rate of decrease of $f(x)$ after it reaches the maximum value levels off. It's also interesting to note that $f(x)$ is clearly an EVEN function, so your derivative graph should be an ODD function.

#4 Definition of Derivative, Limits. **(a)** $1 = \lim_{h\to 0}\frac{e^h-1}{h} = \lim_{h\to 0}\frac{e^h-e^0}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h-0} = \lim_{h\to a}\frac{f(h)-f(a)}{h-a} = f'(a)$. Also, $\lim_{h\to 0}\frac{e^h-1}{h} = \lim_{h\to 0}\frac{e^{0+h}-e^0}{h} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = f'(0)$. Clearly $f(x)=e^x$ and $a=0$ which means that $f'(a)=e^0=1$. **(b)** Use the limit definition of the derivative when $f(x)=e^x$ to show $f'(x)=e^x$. $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f=e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^xe^h-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = \lim_{h\to 0} e^x\cdot\lim_{h\to 0}\frac{(e^h-1)}{h} = e^x\cdot 1 = e^x$

#5 Chain Rule, Tangent Lines, Implicit Differentiation, Logarithmic Differentiation.

(a) Given $x = ye^y$ there are multiple ways to find $\frac{dy}{dx}$. You could find $\frac{dx}{dy}$ and invert the answer.

$$\begin{aligned}
x &= ye^y \\
\frac{d}{dy}[x] &= \frac{d}{dy}[ye^y] \qquad \text{(Differentiate both side with respect to } y\text{)} \\
\frac{dx}{dy} &= \frac{d}{dy}[y]e^y + y\frac{d}{dy}[e^y] \\
&= 1\cdot e^y + y\cdot e^y \\
&= e^y(1+y) \\
\frac{dy}{dx} &= \frac{1}{e^y(1+y)} \\
\frac{dy}{dx} &= \frac{y}{x(1+y)} \qquad \text{Since } x = ye^y \Leftrightarrow x/y = e^y \Leftrightarrow y/x = 1/e^y
\end{aligned}$$

You could use logarithmic differentiation.

$$\begin{aligned}
x &= ye^y \\
\ln(x) &= \ln(ye^y) \qquad \text{(Take the natural log of both sides and THEN implicitly differentiate)} \\
\ln(x) &= \ln(y) + \ln(e^y) \\
&= \ln(y) + y \\
\frac{d}{dx}[\ln(x)] &= \frac{d}{dx}[\ln(y)+y] \qquad \text{(Differentiate both sides now with respect to x)} \\
\frac{1}{x} &= \frac{1}{y}\frac{dy}{dx} + \frac{dy}{dx} \\
&= \left(\frac{1}{y}+1\right)\frac{dy}{dx} \\
&= \left(\frac{y+1}{y}\right)\frac{dy}{dx} \\
\frac{1}{x}\left(\frac{y}{y+1}\right) &= \frac{dy}{dx}
\end{aligned}$$

Or you could use implicit differentiation directly.

$$\begin{aligned}
x &= ye^y \\
\frac{d}{dx}[x] &= \frac{d}{dx}[ye^y] \text{(The square brackets are to show precisely what is begin differentiated)} \\
1 &= \frac{d}{dx}[y]e^y + y\frac{d}{dx}[e^y] \qquad \text{Product Rule !} \\
&= \frac{dy}{dx}e^y + ye^y\frac{dy}{dx} \\
&= e^y(1+y)\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{1}{e^y(1+y)}
\end{aligned}$$

(b) Clearly you can draw the tangent line on the graph in the picture so its tangent line must exist at $(0,0)$. So, instead of using the GIVEN derivative (because it CLEARLY can NOT be evaluated at $x=0, y=0$) you have to change it using the fact that $x = ye^y$ for all x and y to look like $\frac{dy}{dx} = \frac{1}{(y+1)e^y}$ which CAN be evaluated at $x=-0, y=0$. Thus $\left.\frac{dy}{dx}\right|_{x=0,y=0} = 1$ which is the slope of the tangent line. A line with a slope of 1 that goes through the origin is $y = x$ which is the equation of the tangent line.