# Test 1: Basic Calculus I

Math 110 Fall 2007 Thursday September 26

©**Prof. Ron Buckmire** 7:00pm

Name: Key

**Directions**: Read *all* problems first before answering any of them. There are 6 pages in this test. This test is intended to be taken in 55-minutes. You may have as much time as you like (within reason!) **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

**Pledge:** I, ______________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

| No. | Score | Maximum |
|---|---|---|
| 1 | | 30 |
| 2 | | 40 |
| 3 | | 30 |
| BONUS | | 10 |
| **Total** | | **100** |

**1. Logarithms, Exponentials, Piecewise-Defined Functions.** *30 points.*

Although your Calculus textbook is wonderful, it is not comprehensive; there are some items missing from the text. For example, in the section on logarithms and exponentials, the text does not tell you about the little-known Buckmire logarithm function, a.k.a blogarithm function, denoted $\text{blog}(x)$ which is defined as

$$\text{blog}(x) = \begin{cases} \ln(x), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ \ln(-x), & \text{if } x < 0 \end{cases}$$

In this problem I would like you to explore how blogarithms are different (if they are) from natural logarithms.

**(a)** *10 points.* Is $\text{blog}(e^x) = x$ for every value of $x$? PROVE YOUR ANSWER.

$\text{blog}(e^x) = \ln(e^x) = x$ since $e^x > 0$

Since $e^x > 0$, $\text{blog}(e^x) = x$ for all values of $x$.

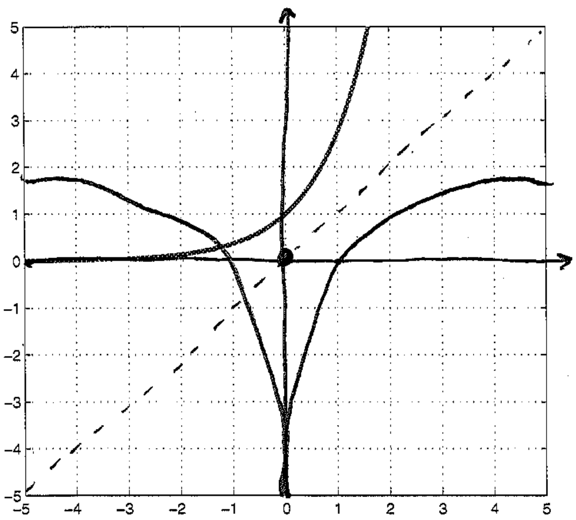
**(b)** *10 points.* Is $e^{\text{blog}(x)} = x$ for every value of $x$? PROVE YOUR ANSWER.

If $x > 0$ $e^{\text{blog}(x)} = e^{\ln(x)} = x$

If $x = 0$ $e^{\text{blog}(0)} = e^0 = 1 \neq 0$

If $x < 0$ $e^{\text{blog}(x)} = e^{\ln(-x)} = -x \neq x$

**(c)** *10 points.* On the axes below, sketch a graph of $\text{blog}(x)$. For your convenience, I have included a graph of the everywhere continuous, invertible function $e^x$.

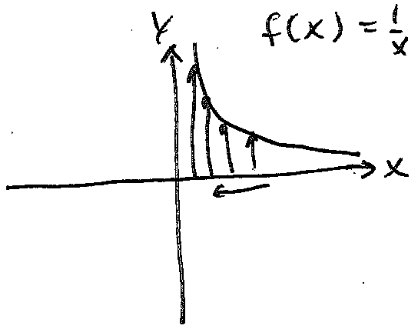


**2. Limits, Continuity.** *40 points.* This problem is where I provide the answers and YOU provide the question. Consider the expression $\lim_{x \to a} f(x) = L$. In each case, I provide you with $L$ and I want YOU to provide a formula for $f(x)$ and an input value $a$ and then show that for the $f(x)$ and $a$ you write down, $\lim_{x \to a} f(x)$ indeed represents the limit scenario I have described. Remember, a picture is worth a thousand words!

(a) (*8 points.*) Write down an example of a one-sided limit to a real number $a$ of a function $f(x)$ where the limit DOES NOT EXIST. Clearly, indicate your choices of $f(x)$ and $a$. Finally, show and explain why the limit you created does not exist.

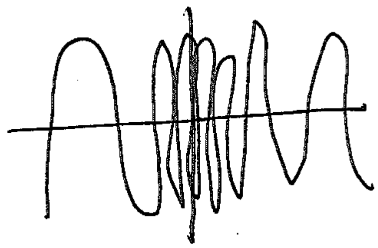
$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$f(x) = \frac{1}{x}$, $a = 0$




OR

$$\lim_{x \to 0^-} \sin\left(\frac{1}{x}\right) = \text{D.N.E}$$



$$\lim_{x \to 0^-} \sin\left(\frac{1}{x}\right) = \lim_{x \to -\infty} \sin(x) = \text{D.N.E.}$$

(b) (*8 points.*) Write down an example of a limit of indeterminate form where the limit actually equals zero. Clearly, indicate your choices of $f(x)$ and $a$. Finally, show and explain why the limit you created equals zero.

$$\lim_{x \to \infty} \frac{x^2+2}{x^3+7} = \frac{\infty}{\infty} \quad \text{"indeterminate form"}$$

$$= \lim_{x \to \infty} \frac{x^2}{x^3} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

OR

$$\lim_{x \to 0} \frac{\sin^2 x}{x} = \frac{0}{0} \quad \text{"indeterminate form"}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \sin x = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \sin x = 1 \cdot 0 = 0$$

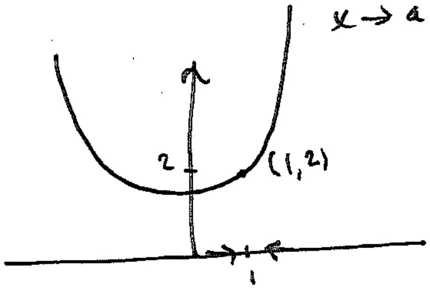
(c) (*8 points.*) Write down an example of a two-sided limit to a real number $a$ of a **non-constant** function $f(x)$ where the limit $L = 2$. Clearly, indicate your choices of $f(x)$ and $a$. Finally, show and explain why the limit you created equals 2.

$$\lim_{x \to 1} x^2 + 1 = 1^2 + 1 = 2 = f(1)$$

Since $f(x)$ is a polynomial, $\lim_{x \to a} f(x) = f(a)$

$f(x) = x^2 + 1$

$a = 1$




(d) (*16 points.*) Write down an example of a function $f(x)$ which has a removable discontinuity at $x = 0$. Show that your function is indeed discontinuous at $x = 0$ and then write down a new function $g(x)$ which is continuous at $x = 0$. Finally, show and explain why your new function $g(x)$ is continuous at $x = 0$.

$f(x) = \frac{x^2}{x}$, $f(0)$ is not defined (division by zero)

$$\lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0$$

$$g(x) = \begin{cases} \frac{x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

OR

$f(x) = \frac{\sin(x)}{x}$, $f(0)$ is not defined (division by zero)

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

$$g(x) = \begin{cases} \frac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

**3. Functions, Inverses.** *30 points.*

Consider the table representing the function $f(x)$ on its entire domain, the set $\{-2,-1,0,1,2\}$.

| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | -2 | 1 | 2 | -1 | 0 |
| $f(f(x))$ | -2 | -1 | 0 | 1 | 2 |
| $f(f(f(x)))$ | -2 | 1 | 2 | -1 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

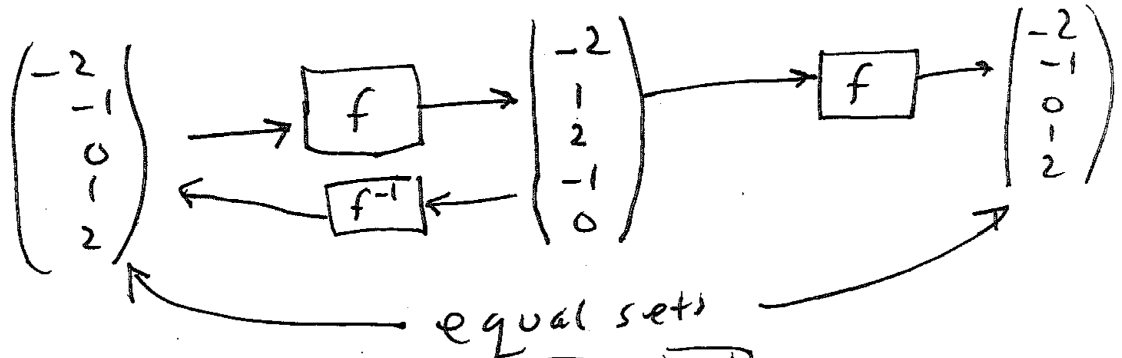
**(a)** (*20 points.*) Fill in the blank spaces in the table with the output values for $f$ composed with itself once, i.e. $f^2(x)$ and $f$ composed with itself twice, i.e. $f^3(x)$ corresponding to the input value in each column.

**(b)** (*10 points.*) An **involutory** function is a function whose inverse is equal to itself. Is the function $f(x)$ represented by the above table an involutory function? Please provide evidence with carefully explained reasoning to support your answer to this question.

$f^2(x) = f(f(x)) = x$, which means $f = f^{-1}$

The third line equals the first line.

Yes, $f(x)$ is involutory since it is equal to its own inverse.




so $\boxed{f} = \boxed{f^{-1}}$

**BONUS QUESTION.** *10 points.* **CHOOSE ONE.** Suppose you continue the process of composing $f(x)$ with itself over and over again without end. Obtain an expression for $f^n(x)$, i.e. $f$ composed with itself $n-1$ times in a row. If you consider the sequence of functions $f(x)$, $f^2(x)$, $f^3(x)$, $\ldots$, $f^n(x)$, $\ldots$ do you think the sequence has a limit? In other words, what is $\lim\limits_{n\to\infty} f^n(x)$? EXPLAIN YOUR ANSWER.

OR

In a short essay (2-3 paragraphs), Discuss whether the blogarithm function is an invertible, everywhere continuous function. Be very careful to fully explain the meanings of the terms invertible and everywhere continuous in order to prove your point as to whether the blogarithm function possesses these two properties.

$$f^n(x) = \begin{cases} f(x), & \text{when } n \text{ is odd} \\ x, & \text{when } n \text{ is even} \end{cases}$$ where $x = \{-2, -1, 0, 1, 2\}$

Thus $\lim\limits_{n\to\infty} f_n(x)$ Does Not Exist because I can't write down a specific function $F(x)$ that $\lim\limits_{n\to\infty} f^n(x)$ gets closer to. $f^n(x)$ oscillates between the two sets.

Sadly, the blogarithm function is NOT invertible since it FAILS the horizontal line test, and is thus not one-to-one [two different ~~output~~ values can ~~p~~produce the same output, i.e. $blog(1) = blog(-1) = 0$]

The blogarithm function is continuous everywhere except $x = 0$.

$\lim\limits_{x\to 0^-} blog(x) = -\infty = \lim\limits_{x\to 0^+} blog(x)$

but $blog(0) = 0$. Since the $\lim\limits_{x\to 0} blog(x)$ does not exist $blog(x)$ is NOT continuous at $x=0$