MATH 110 Final Exam

December 13, 2004

Name: 

Please write up your solutions in the space provided. You may use the back of the pages for scratch work and for extra space if you need it. If you wish me to grade any work on the backs of pages, please draw attention to it on the front pages. Boxed answers are appreciated for numerical problems. A calculator and two pages of notes are permitted for this exam.

Be sure to carefully read the instructions for each problem.

There are 7 problems worth 74 points.

Good luck!

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<tr>
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1. Let \( f(x) = x^3 + 5x + 2 \).

(a) Explain why it is true that \( f \) has an inverse function \( f^{-1} \).

\[
f'(x) = 3x^2 + 5 > 0 \quad \text{for all } x \in \mathbb{R}
\]

Since \( f' > 0 \), \( f \) is monotonically increasing and must be a one-to-one function.

Since \( f \) is one-to-one, it is invertible. Thus \( f^{-1} \) exists.

(b) Sketch the graph of \( f \) and \( f^{-1} \) on the same set of axes.

(c) Find the equation of the tangent line to \( f^{-1} \) at \( x = 2 \). This is ambiguous!

\[
\begin{align*}
\text{When } x = 2, & \quad f(2) = 2^3 + 5(2) + 2 \\
& = 8 + 10 + 2 = 20 = y
\end{align*}
\]

\[
\Rightarrow f^{-1}(20) = 2
\]

\[
\text{When } x = 2, \quad \frac{dy}{dx} \bigg|_{x=2} = 3x^2 + 5 \bigg|_{x=2} = 3(2)^2 + 5 = 17
\]

The point on \( f^{-1}(y) \) is \((20,2)\).

Tangent line is

\[
\begin{align*}
Y - 2 &= \frac{1}{17} (x - 20) \\
Y &= \frac{x}{17} - \frac{20}{17} + \frac{2}{17} = \frac{x + 14}{17}
\end{align*}
\]

\[
\text{If } \text{they mean } Y = f^{-1}(x) \text{ and tangent line at } x = 2, \\
f^{-1}(2) = \theta \Rightarrow 2 = f(\theta) \\
(2,0) \text{ is on } \Rightarrow \theta = 0 \\
\text{curve } v = \frac{1}{17}(x-2) \Rightarrow \frac{v}{x} = \frac{14}{17} - \frac{17}{x}.
\]
2. (a) Find the exact value of \( \tan(\cos^{-1}(\frac{4}{5})) \).

\[
\cos^{-1}(\frac{4}{5}) = \theta \\
\tan \theta = \tan \left( \cos^{-1}(\frac{4}{5}) \right) \\
\frac{3}{4} = \tan \left[ \cos^{-1}(\frac{4}{5}) \right]
\]

(b) Simplify \( \cot(\cos^{-1} x) \). For what values of \( x \) does the simplification hold?

\[
y = \cos^{-1} x \\
\Rightarrow \cos y = x \\
\cot(y) = \frac{\cos(y)}{\sin(y)} \\
\cot(\cos^{-1}(x)) = \frac{x}{\sqrt{1 - x^2}}
\]

\( |x| < 1 \) or \(-1 < x < 1\) for this expression to be valid.
3. Choose only one of the following problems to solve.

(a) If the half-life of radium is 1690 years, what percent of the initial amount will remain after 100 years? This problem involves exponential growth or decay.

(b) What is the volume of the largest box you can make by cutting corners from an 8.5-inch by 11-inch piece of paper and folding up the sides?

\[
\frac{dR}{dt} = -\lambda R, \quad R(0) = R_0 \quad \text{(Exponential Decay)}
\]

\[
R(t) = R_0 e^{-\lambda t} \quad R(1690) = \frac{1}{2} R_0
\]

\[
\frac{1}{2} R_0 = R_0 e^{-\lambda 1690}
\]

\[
\frac{1}{2} = e^{-1690 \lambda}
\]

\[
\ln \left( \frac{1}{2} \right) = \ln \left( e^{-1690 \lambda} \right) = -1690 \lambda
\]

\[
-\ln 2 = -1690 \lambda
\]

\[
\frac{-\ln 2}{1690} = \lambda
\]

\[
R(100) = R_0 e^{-100 \lambda} = R_0 e^{\frac{-100 \ln 2}{1690}} = \text{Fraction of initial amount left}
\]

\[
\frac{\text{Volume left}}{R_0} = R(100) \times 100 = 100 e^{\frac{-100 \ln 2}{1690}} = 95.98 \%
\]

\[
\text{Volume} = (11 - 2x)(8.5 - 2y) x
\]

\[
x = y \text{ in order to have even box}
\]

\[
V = 24x^2 - 7.8
\]

\[
V'(1.59) = -39.95 \quad \text{max}
\]

\[
V''(1.91) = +39.95 \quad \text{min}
\]

\[
V = 12x^2 - 7.8x - 9.35 = 0
\]

\[
x = \frac{78 \pm \sqrt{78^2 - 4 \cdot 12 \cdot 93.5}}{24}
\]
4. Let \( f(x) = \frac{1}{1 + x^2} \).

**In order to receive full credit on this problem, you must derive your answers using techniques learned in this course and your answers must be exact.**

(a) Find all (if any) relative extrema of \( f \). If \( f \) has no relative extrema, explain why.

\[
\begin{align*}
\frac{d}{dx} f(x) &= -\frac{2x}{(1 + x^2)^2} \\
&= \frac{0 \cdot (1 + x^2) - (2x) \cdot 1}{(1 + x^2)^2} \\
&= \frac{-2x}{(1 + x^2)^2} \\
\therefore x &= 0 \quad \Rightarrow \quad \frac{df}{dx} \text{ does not exist for } x \neq 0 \quad \text{No places where } f' \text{ ONE since } x^2 + 1 > 0 \\
&\text{for all } x
\end{align*}
\]

\[\begin{array}{c|c|c|c}
\hline
x & f' & f & \text{Local Max at (0, 1)} \\
\hline
0 & + & + & \\
\hline
\end{array}\]

(b) Find all (if any) points of inflection of \( f \). If \( f \) has no inflection points, briefly explain why.

\[
\begin{align*}
f''(x) &= -\frac{2(1 + x^2)^2 - (-2x) \cdot 2(1 + x^2) \cdot 2x}{(1 + x^2)^4} \\
&= -\frac{2(1 + x^2) + 2 \cdot 2 \cdot 2x}{(1 + x^2)^3} \\
&= -\frac{2 - 2x^2 + 8x^2}{(1 + x^2)^3} \\
&= \frac{6x^2 - 2}{(1 + x^2)^3} \\
&= \frac{2(3x^2 - 1)}{(1 + x^2)^3}
\end{align*}
\]

\( \therefore x = \frac{\pm 1}{\sqrt{3}} \)

Problem continues on next page...
(c) Sketch a graph of \( f \). Identify all significant features of the graph including all local and absolute extrema, inflection points and asymptotes.

\[
\lim_{x \to -\infty} \frac{1}{1 + x^2} = 0 \quad \lim_{x \to \infty} \frac{1}{1 + x^2} = 0
\]

(d) Find the exact point on the graph of \( f \) where the slope attains its maximum.

The slope of \( f \) attains its maximum where \( f'' \) switches from +ve (concave up) to -ve (concave down), this happens at the inflection point \((\frac{-1}{\sqrt{3}}, \frac{3}{4})\).
5. Consider the initial value problem

\[ y' = \frac{3x}{y}, \quad y(1) = 1. \]

(a) Perform two iterations of Euler's method with step size \( h = 0.1 \). (That is, compute \( y_1 \) and \( y_2 \).) What value does \( y_2 \) approximate?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( y' )</th>
<th>( \Delta y )</th>
<th>( \Delta y \approx y' \cdot \Delta x )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>3</td>
<td>0.3</td>
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<tr>
<td>1.1</td>
<td>1.3</td>
<td>( \frac{3 \cdot (1.1)}{1.3} ) = ( \frac{3.3}{1.3} )</td>
<td>0.254</td>
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<tr>
<td>1.2</td>
<td>1.534</td>
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(b) How many iterations of Euler's method are necessary to approximate \( y(3) \) with \( h = 0.1 \)?

\[ 3 - 1 = 2 = \text{difference between starting & ending points} \]

\[ 2 = N \cdot \Delta x \]

\[ 2 = N \cdot 0.1 \]

\[ 2 \div 0.1 = 20 \]

\( 20 = N \)

Take 20 iterations or steps of Euler's Method to approx \( y(3) \)
6. Let \( f(x) = \frac{1 - e^x}{x} \).

(a) Explain why \( f \) is continuous at every point except \( x = 0 \).

The ratio of continuous functions is continuous except where the denominator will output zero.

(b) Explain why the discontinuity at \( x = 0 \) is removable.

\[
\lim_{x \to 0} \frac{1 - e^x}{x} = \lim_{x \to 0} \frac{-e^x}{1} = -1
\]

Since the limit exists as \( x \to 0 \), the discontinuity is removable.

(c) Define \( f(0) \) so that \( f \) is continuous on the entire real line.

\[
f(0) = -1
\]

(d) With \( f(0) \) defined as in (c), show that \( f \) is differentiable at 0 and compute \( f'(0) \).

\[
f'(x) = \begin{cases} -1, & x < 0 \\ \frac{1 - e^x}{x}, & x > 0 \end{cases}
\]

\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{1 - e^x}{x} - (-1)
\]

\[
= \lim_{x \to 0} \frac{1 - e^x + 1}{x} = \lim_{x \to 0} \frac{1 - e^x + x}{x}
\]

\[
= \lim_{x \to 0} \frac{1 - e^x + x}{x} = \lim_{x \to 0} \frac{-e^x}{x^2} = \lim_{x \to 0} \frac{-e^x}{2x} = -\frac{1}{2}
\]
7. Consider the function \( f(x) = \tan^{-1}(x^2) = \arctan(x^2) \)

(a) Find a linear approximation of \( f \) that is valid for values of \( x \) near 1.

\[
\frac{f'(x)}{1 + (x^2)^2} \cdot 2x = \frac{2x}{1 + x^4}
\]

\[
f'(1) = \frac{2}{1 + 1} = 1
\]

\[
f(1) = \arctan(1) = \frac{\pi}{4}
\]

\[
f(x) \approx f(1) + f'(1)(x - 1) = \frac{\pi}{4} + 1(x - 1)
\]

(b) Use the linear approximation in (a) to estimate the value \( \tan^{-1}(2.25) \). [Hint: What value of \( x \) gives \( f'(x) = \tan^{-1}(2.25) \)?]

\[
\text{If } x = 1.5
\]

\[
f(1.5) = \tan^{-1}(1.5^2) = \tan^{-1}(2.25)
\]

\[
f(1.5) \approx \frac{\pi}{4} + 1(1.5 - 1)
\]

\[
\approx \frac{\pi}{4} + \frac{1}{2} \approx 1.285
\]