MATH 110 Final Exam

December 13, 2004

Name: Key

Please write up your solutions in the space provided. You may use the back of the pages for scratch work and for extra space if you need it. If you wish me to grade any work on the backs of pages, please draw attention to it on the front pages. Boxed answers are appreciated for numerical problems. **A calculator and two pages of notes are permitted for this exam.**

Be sure to carefully read the instructions for each problem.

There are 7 problems worth 74 points.

Good luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	12	
5	10	
6	12	
7	10	
Total	74	

1. Let $f(x) = x^3 + 5x + 2$.

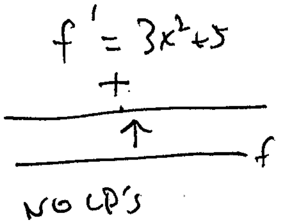
(a) Explain why it is true that f has an inverse function f^{-1}.

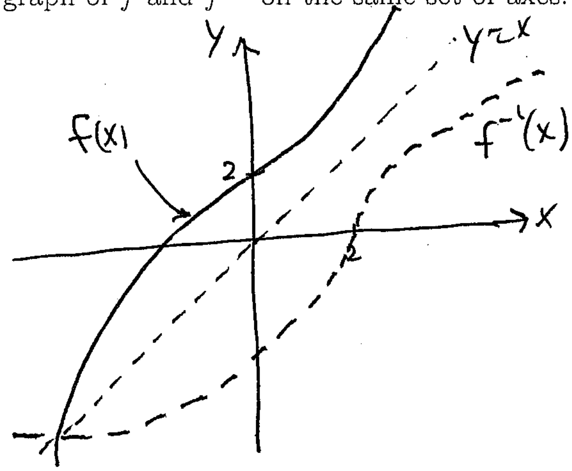
$f' = 3x^2 + 5 > 0$ for all $x \in \mathbb{R}$

Since $f' > 0$, f is monotonically increasing and must be a one-to-one function.

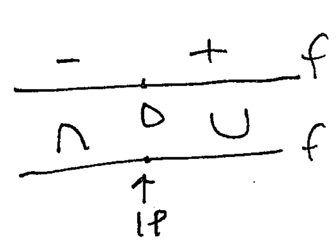
Since f is one-to-one, it is invertible. Thus f^{-1} exists.

(b) Sketch the graph of f and f^{-1} on the same set of axes.





$f'' = 6x$



(c) Find the equation of the tangent line to f^{-1} at $x = 2$. This is ambiguous!

When $x = 2$, $f(2) = 2^3 + 5(2) + 2$
$= 8 + 10 + 2 = 20 = y$

$y = x^3 + 5x + 2$

If you Assume $f^{-1}(y) = x$

$\Rightarrow f^{-1}(20) = 2$

when $x = 2$, $\left.\frac{dy}{dx}\right|_{x=2} = 3x^2 + 5\Big|_{x=2} = 3 \cdot 2^2 + 5 = 17$

$\left.\frac{dx}{dy}\right|_{x=2} = \frac{1}{\left.\frac{dy}{dx}\right|_{x=2}} = \frac{1}{17}$ = slope of tangent line

The point on $f^{-1}(y)$ is $(20, 2)$

Tangent line is

$y - 2 = \frac{1}{17}(x - 20)$

$y = \frac{x}{17} - \frac{20}{17} + \frac{34}{17} = \boxed{\frac{x}{17} + \frac{14}{17} = y}$

OR If they mean $y = f^{-1}(x)$ and tangent line at $x = 2$.

$f^{-1}(2) = \heartsuit \Rightarrow 2 = f(\heartsuit)$

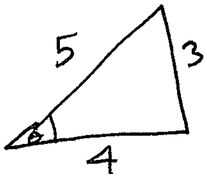
$2 = \heartsuit^3 + 5\heartsuit + 2$

$(2, 0)$ is on curve

$\Rightarrow \heartsuit = 0$

$y - 0 = \frac{1}{17}(x - 2) \Rightarrow \boxed{y = \frac{x}{17} - \frac{2}{17}}$

2. (a) Find the exact value of $\tan(\cos^{-1}(\frac{4}{5}))$.



$$\cos^{-1}\left(\frac{4}{5}\right) = \theta$$

$$\tan\theta = \tan\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$$

$$\frac{3}{4} = \tan\left[\cos^{-1}\left(\frac{4}{5}\right)\right]$$

(b) Simplify $\cot(\cos^{-1} x)$. For what values of x does the simplification hold?

$$y = \cos^{-1} x$$

$$\Rightarrow \cos y = x$$

$$\cot(\cos^{-1}(x)) = \cot(y) = \frac{\cos(y)}{\sin(y)}$$

$$= \frac{x}{\sqrt{1-\cos^2(y)}}$$

$$\cot(\cos^{-1}(x)) = \frac{x}{\sqrt{1-x^2}}$$

$|x| < 1$ or $-1 < x < 1$ for this expression to be valid.

3. Choose **only one** of the following problems to solve.

(a) If the half-life of radium is 1690 years, what percent of the initial amount will remain after 100 years? *This problem involves exponential growth or decay.*

(b) What is the volume of the largest box you can make by cutting corners from an 8.5-inch by 11-inch piece of paper and folding up the sides?

a. $\frac{dR}{dt} = -\lambda R$, $R(0) = R_0$ (Exponential Decay)

$R(t) = R_0 e^{-\lambda t}$ $\quad$ $R(1690) = \frac{1}{2} R_0$

$\frac{1}{2} R_0 = R_0 e^{-\lambda 1690}$

$\frac{1}{2} = e^{-1690\lambda}$

$\ln\left(\frac{1}{2}\right) = \ln\left(e^{-1690\lambda}\right) = -1690\lambda$

$-\ln 2 = -1690\lambda$

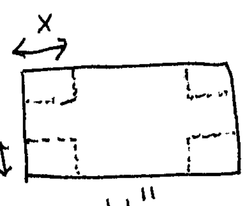
$\frac{\ln 2}{1690} = \lambda$

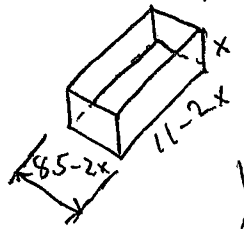
$R(100) = R_0 e^{-\lambda 100} = R_0 e^{-\frac{100}{1690}\ln 2}$ = Fraction of initial amount left

$\%$ left $= \frac{R(100)}{R_0} \times 100 = 100 e^{-\frac{100 \ln 2}{1690}} = \cancel{0.98}$
$= 95.98\%$

b.

$\boxed{\text{Dimensions of box } 7.82 \times 5.32 \times 1.59}$





$\boxed{\text{Volume} = 66 \text{ in}^3}$

Volume $= (11-2x)(8.5-2y)x$

$x = y$ in order to have even box

$V(x) = (11-2x)(8.5-2x)x$

$= 93.5x - 17x^2 - 22x^2 + 4x^3$

$= 4x^3 - 39x^2 - 93.5x$

$V' = 12x^2 - 78x - 93.5 = 0$

$x = \frac{78 \pm \sqrt{78^2 - 4 \cdot 12 \cdot 93.5}}{24}$

$x = \frac{78 \pm \sqrt{1596}}{24}$

$x \approx 4.91$ or 1.59

$V'' = 24x - 78$

$V''(1.59) = -39.95 \leftarrow$ MAX

$V''(4.91) = +39.95 \leftarrow$ MIN

4. Let $f(x) = \dfrac{1}{1+x^2}$.

In order to receive full credit on this problem, you must derive your answers using techniques learned in this course and your answers must be exact.

(a) Find all (if any) relative extrema of f. If f has no relative extrema, explain why.

$$f'(x) = \frac{-2x}{(1+x^2)^2} \quad \text{(Reciprocal Rule)}$$

$$= \frac{0\cdot(1+x^2) - (2x)\cdot 1}{(1+x^2)^2} \quad \text{(Quotient Rule)}$$

$$0 = \frac{-2x}{(1+x^2)^2} \Rightarrow x = 0$$

No places where f' DNE since $x^2+1>0$ for all x

$+ \quad - \quad f'$

$\uparrow \quad 0 \quad \downarrow \quad f$

0

$x=0$ is a LOCAL MAX at $(0,1)$

(b) Find all (if any) points of inflection of f. If f has no inflection points, briefly explain why.

$$f''(x) = \frac{-2\cdot(1+x^2)^2 - (-2x)\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4}$$

$$= \frac{-2(1+x^2) + 2x\cdot 2\cdot 2x}{(1+x^2)^3}$$

$$f'' = \frac{-2-2x^2+8x^2}{(1+x^2)^3} = \frac{6x^2-2}{(1+x^2)^3} = \frac{2(3x^2-1)}{(1+x^2)^3}$$

$$0 = 3x^2-1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

Problem continues on next page.— — — >

$$f\left(-\tfrac{1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{4/3} = \frac{3}{4}$$

$+ \quad - \quad + \quad f''$

$-\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}}$

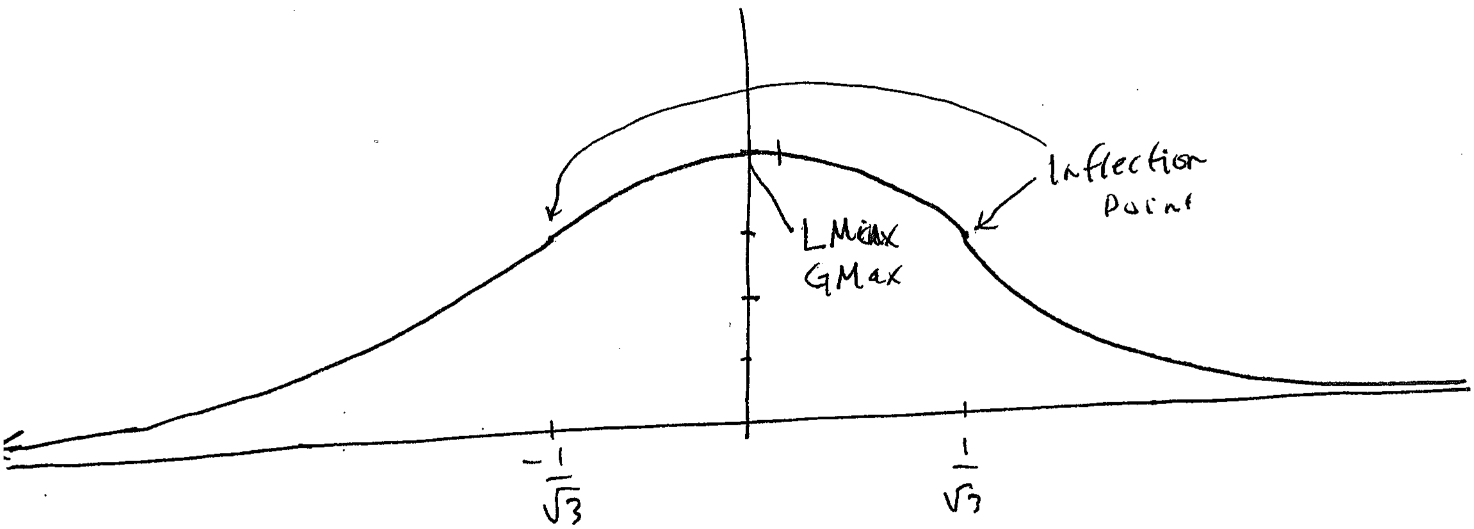
$\cup \quad \cap \quad \cup$

IP IP

Inflection Points at $\left(-\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$

(c) Sketch a graph of f. Identify all significant features of the graph including all local and absolute extrema, inflection points and asymptotes.

$$\lim_{x \to -\infty} \frac{1}{1+x^2} = 0 \qquad \lim_{x \to \infty} = \frac{1}{1+x^2} = 0$$



(d) Find the exact point on the graph of f where the slope attains its maximum.

The slope of f attains its maximum where f'↑ the fastest and that is where f' has a max which is where f'' switches from +ve (concave up) to −ve (concave down), this happens at the inflection point $\left(-\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$.

5. Consider the initial value problem

$$y' = \frac{3x}{y}, \quad y(1) = 1.$$

(a) Perform two iterations of Euler's method with step size $h = 0.1$. (That is, compute y_1 and y_2.) What value does y_2 approximate?

$\Delta y \approx y' \cdot \Delta x$

x	y	y'	Δy
1	1	3	0.3
1.1	1.3	$\frac{3\cdot(1.1)}{1.3} = \frac{33}{13}$	0.254
1.2	1.554		

(b) How many iterations of Euler's method are necessary to approximate $y(3)$ with $h = 0.1$?

$3 - 1 = 2$ = difference between starting & ending points

$$2 = N \cdot \Delta x$$

$$2 = N \cdot 0.1$$

$$\frac{2}{0.1} = N$$

$$20 = N$$

Takes 20 iterations or steps of Euler's Method to approx $y(3)$

6. Let $f(x) = \dfrac{1-e^x}{x}$.

(a) Explain why f is continuous at every point **except** $x = 0$.

The ratio of continuous functions is continuous except where the denominator will output zero.

(b) Explain why the discontinuity at $x = 0$ is removable.

$$\lim_{x \to 0} \frac{1-e^x}{x} \overset{L'H}{=} \lim_{x \to 0} \frac{-e^x}{1} = -1$$

Since the limit ~~x →~~ exists as ~~one app~~. at $x=0$, the discontinuity is removable

(c) Define $f(0)$ so that f is continuous on the entire real line.

$$f(0) = -1$$

(d) With $f(0)$ defined as in (c), show that f is differentiable at 0 and compute $f'(0)$.

$$f(x) = \begin{cases} -1, & x = 0 \\ \dfrac{1-e^x}{x}, & x \neq 0 \end{cases}$$

$$\boxed{f'(0) = -\frac{1}{2}}$$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{1-e^x}{x} - (-1)}{x}$$

$$= \lim_{x \to 0} \frac{\frac{1-e^x}{x} + 1}{x} = \lim_{x \to 0} \frac{\frac{1-e^x}{x} + \frac{x}{x}}{x}$$

$$= \lim_{x \to 0} \frac{\frac{1-e^x+x}{x}}{x} = \lim_{x \to 0} \frac{1-e^x+x}{x^2} \overset{L'H}{=} \lim_{x \to 0} \frac{1-e^x}{2x} \overset{LH}{=} \lim_{x \to 0} \frac{-e^x}{2}$$

$$= -1/2$$

7. Consider the function $f(x) = \tan^{-1}(x^2)$. $= \arctan(x^2)$

(a) Find a linear approximation of f that is valid for values of x near 1.

$$f'(x) = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$$

$$f'(1) = \frac{2}{1+1} = 1$$

$$f(1) = \arctan(1) = \pi/4$$

$$f(x) \approx \underbrace{f(1) + f'(1)(x-1)}_{\text{tangent line at } x=1} = \frac{\pi}{4} + 1(x-1)$$

(b) Use the linear approximation in (a) to estimate the value $\tan^{-1}(2.25)$. [Hint: What value of x gives $f(x) = \tan^{-1}(2.25)$?]

If $x = 1.5$

$$f(1.5) = \tan^{-1}(1.5^2) = \tan^{-1}(2.25)$$

$$f(1.5) \approx \frac{\pi}{4} + 1(1.5-1)$$

$$\approx \frac{\pi}{4} + \frac{1}{2} \approx 1.285$$