MATH 110
EXAM 3 REVIEW EXERCISES SOLUTIONS

1. $y = \arctan(x) \Rightarrow x = \tan(y)$

$$\frac{dx}{dy} = \sec^2(y) = 1+\tan^2(y) = 1+x^2$$

$$\frac{dx}{dy} = 1+x^2 \qquad \frac{dy}{dx} = \frac{1}{1+x^2}$$

2. $y = x^x$

To show $y = x^x$ satisfies the DE & IC

$x = 2,\ y = 4$

$y = 2^2 = 4$ ✓

$$y' = \frac{d}{dx}(x^x) = \frac{d}{dx}\left(e^{\ln(x^x)}\right) = \frac{d}{dx} e^{x\ln x} = e^{x\ln x}\cdot (x\ln x)'$$

$$= x^x\left(x\cdot\frac{1}{x} + 1\cdot\ln x\right)$$

$$= x^x(1+\ln(x))$$

$$y' = x^x + x^x\ln x$$

$$y' = y + \frac{y\ln(y)}{x}$$

$$y = x^x$$
$$\ln y = x\ln x$$
$$y\ln y = x^x x\ln x$$
$$\frac{y\ln y}{x} = \frac{x^{x+1}\ln x}{x}$$
$$\frac{y\ln y}{x} = x^x\ln(x)$$

3. (a) $\displaystyle \lim_{x\to\infty}\frac{\ln(x)}{\sqrt{x}} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty}\frac{2\sqrt{x}}{x} = 0$

(b) $\displaystyle \lim_{x\to 0}\frac{\tan(x)-x}{x^3} \overset{L'H}{=} \lim_{x\to 0}\frac{\sec^2(x)-1}{3x^2} \overset{L'H}{=} \lim_{x\to 0}\frac{2\sec(x)\sec(x)\tan(x)}{6x}$

$\displaystyle = \lim_{x\to 0}\frac{\sec^3(x)}{3}\ \lim_{x\to 0}\frac{\sin(x)}{x} = \frac{1}{3}\cdot 1 = \frac{1}{3}$

3(c) $\lim_{x\to 0^+} \frac{\sqrt{x}}{\ln(x)} = \text{"}\frac{0}{-\infty}\text{"} = 0$

(d) $\lim_{x\to 0^+} (1-\ln x)^x$

$= \lim_{x\to 0^+} e^{\ln[(1-\ln x)^x]} = e^{\lim_{x\to 0^+} x\ln(1-\ln x)} = e^P$

$P = \lim_{x\to 0^+} x \ln(1-\ln(x)) = \lim_{x\to 0^+} \frac{\ln(1-\ln x)}{1/x} = \text{"}\frac{\infty}{\infty}\text{"}$

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1-\ln(x)}\cdot\frac{-1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} \frac{x}{1-\ln(x)} = \text{"}\frac{0}{\infty}\text{"} = 0$

4. $y = x^{\arctan(x)} = e^{\ln(x^{\arctan x})} = e^{\arctan(x)\ln(x)}$

$y' = e^{\arctan(x)\ln(x)} \cdot (\arctan(x)\ln(x))'$

$= x^{\arctan(x)} [(\arctan x)' \ln(x) + \arctan(x)(\ln x)']$

$= x^{\arctan(x)} \left[\frac{1}{1+x^2}\cdot\ln(x) + \arctan(x)\cdot\frac{1}{x}\right]$

5. $f(x) = \frac{x^3}{x-1}$

$f'(x) = \frac{(x^3)'(x-1) - x^3(x-1)'}{(x-1)^2} = \frac{3x^2(x-1)-x^3(1)}{(x-1)^2}$

$= \frac{3x^3-3x^2-x^3}{(x-1)^2} = \frac{2x^3-3x^2}{(x-1)^2} = \frac{(2x-3)x^2}{(x-1)^2}$

C.P.'s where $f'=0$ or f' DNE

$0 = (2x-3)(x^2)$ or $(x-1)^2 = 0$

$x = \frac{3}{2}$, $x=0$ or $x=1$

3/2 is at a local MIN.

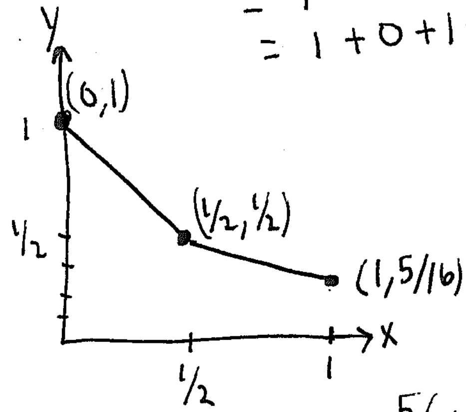
$f''(x) = \frac{(6x^2-6x)(x-1)^2-(2x^3-3x^2)2(x-1)}{(x-1)^4} = \frac{6x(x-1)^2-2x^2(2x-3)}{(x-1)^3}$

$f''(0)=0$ $f''(1)$ DNE $f''(3/2) = \frac{6\cdot 3/2\cdot(1/2)^2}{(1/2)^3} = \frac{9}{1/2} = 18 > 0 \Rightarrow$ MIN

7. ~~6~~. $y(0) = 1, \quad \frac{dy}{dx} = xy^2 - y$

$\Delta x = 0.5$

x	y	y'	Δy
0	1	-1	$-1/2$
$1/2$	$1/2$	$-3/8$	$-3/16$
1	$5/16$		

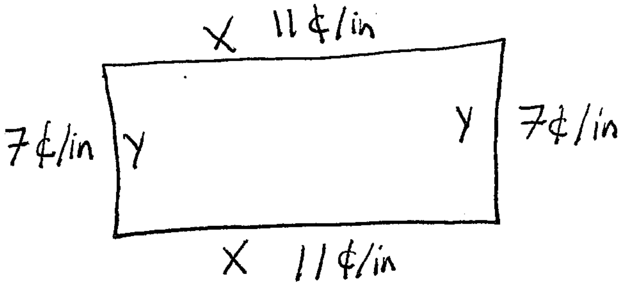


$$y'' = y^2 + x\,2yy' - y'$$
$$= 1^2 + 0\cdot2\cdot1\cdot(-1) - (-1)$$
$$= 1 + 0 + 1 = 2 > 0$$

When y is concave up, Euler's Method produces an underestimate

$5/16$ is less than the actual value

6.

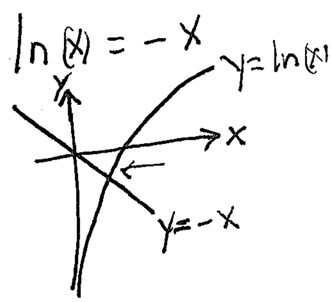


$$A = xy = 100$$
$$C = 2x(11) + 2y(7)$$
$$= 22x + 14y$$
$$C = 22x + 14\left(\frac{100}{x}\right)$$
$$\frac{dC}{dx} = 22 - \frac{1400}{x^2} = 0 \qquad x^2 = \frac{1400}{22} = \frac{700}{11}$$
$$x = 10\sqrt{\frac{7}{11}} \qquad y = \frac{100}{10\sqrt{\frac{7}{11}}} = 10\sqrt{\frac{11}{7}}$$

8. $f(x) = x\ln x - x + \frac{1}{2}x^2$

$f'(x) = \ln x + x\cdot\frac{1}{x} - 1 + x = \ln(x) + x = 0 \Rightarrow \ln(x) = -x$

$f''(x) = \frac{1}{x} + 1$



9. $\dfrac{dy}{dx} = -y^2$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-y^2) = -2y\frac{dy}{dx} = -2y(-y^2) = 2y^3$$

When $y>0$, $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} > 0$

i.e $y>0$, $y\downarrow$ and y concave up

10. "If f is differentiable on (a,b) AND if f has an absolute extremum on (a,b) THEN it must occur at a stationary point of f."

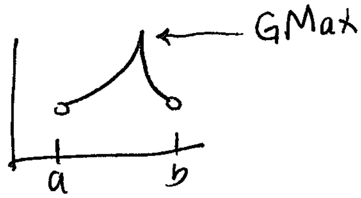
This is TRUE.

On an open interval an absolute extremum will occur at a critical point of f.

Since f' exists at every point in (a,b) the extremum must occur where $f'=0$, i.e. a stationary point

"If is continuous on (a,b) AND if f has an absolute extremum on (a,b) THEN it must occur at a stationary point of f"

FALSE.



The extremum must occur at a critical point of f, but f' may not exist there!

11. $S = x^2 + \frac{1}{x^2}$, $x \in (0, \infty)$

$$\frac{dS}{dx} = 2x - \frac{2}{x^3} = 0$$

$$2x\left(1 - \frac{1}{x^2}\right) = 0 \qquad x = 0 \text{ or } x^2 = 1$$
$$x = +1, -1$$

Only $x = 1$ is in the domain. Only 1 critical point.

$$\frac{d^2S}{dx^2} = 2 + \frac{6}{x^4}$$

$$S''(1) = 2 + \frac{6}{1^4} = 2 + 6 = 8 > 0$$

So At $x = 1$, $S(x)$ has a local min.

Check endpoints of domain to find abs max and min

$$\lim_{x \to 0^+} x^2 + \frac{1}{x^2} = +\infty$$

$$\lim_{x \to \infty} x^2 + \frac{1}{x^2} = +\infty$$

there is no absolute max.

Since $S \to \infty$ $x = 1$ is both local min and absolute min.