11.3 EULER'S METHOD

In the preceding section we saw how to sketch a solution curve to a differential equation using its slope field. In this section we compute points on a solution curve numerically using *Euler's method.* (Leonhard Euler was an eighteenth-century Swiss mathematician.) In Section 11.4 we find formulas for some solution curves.

Here's the concept behind Euler's method. Think of the slope field as a set of signposts directing you across the plane. Pick a starting point (corresponding to the initial value), and calculate the slope at that point using the differential equation. This slope is a signpost telling you the direction to take. Head off a small distance in that direction. Stop and look at the new signpost. Recalculate the slope from the differential equation, using the coordinates of the new point. Change direction to correspond to the new slope, and move another small distance, and so on.

Example 1 Use Euler's method for $dy/dx = y$. Start at the point $P_0 = (0, 1)$ and take $\Delta x = 0.1$.

Solution The slope at the point $P_0 = (0, 1)$ is $dy/dx = 1$. (See Figure 11.25.) As we move from P_0 to P_1, y increases by Δy, where

$$\Delta y = (\text{slope at } P_0)\Delta x = 1(0.1) = 0.1.$$

So we have

$$y\text{-value at } P_1 = (y \text{ value at } P_0) + \Delta y = 1 + 0.1 = 1.1.$$

Table 11.2 *Euler's method for $dy/dx = y$, starting at $(0, 1)$*

	x	y	$\Delta y = (\text{Slope})\Delta x$
P_0	0	1	$0.1 = (1)(0.1)$
P_1	0.1	1.1	$0.11 = (1.1)(0.1)$
P_2	0.2	1.21	$0.121 = (1.21)(0.1)$
P_3	0.3	1.331	$0.1331 = (1.331)(0.1)$
P_4	0.4	1.4641	$0.14641 = (1.4641)(0.1)$
P_5	0.5	1.61051	$0.161051 = (1.61051)(0.1)$

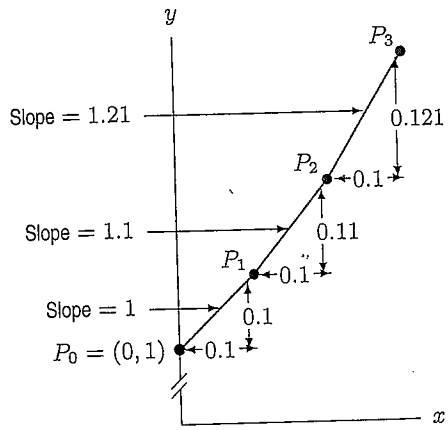

Figure 11.25: Euler's approximate solution to $dy/dx = y$

Thus the point P_1 is $(0.1, 1.1)$. Now, using the differential equation again, we see that

$$\text{Slope at } P_1 = 1.1,$$

so if we move to P_2, then y changes by

$$\Delta y = (\text{slope at } P_1)\Delta x = (1.1)(0.1) = 0.11.$$

This means

$$y\text{-value at } P_2 = (y \text{ value at } P_1) + \Delta y = 1.1 + 0.11 = 1.21.$$

Thus P_2 is $(0.2, 1.21)$. Continuing gives the results in Table 11.2.

Since the solution curves of $dy/dx = y$ are exponentials, they are concave up and bend upward away from the line segments of the slope field. Therefore, in this case, Euler's method produces y-values which are too small.

Notice that Euler's method calculates approximate y-values for points on a solution curve; it does not give a formula for y in terms of x.

Example 2 Show that Euler's method for $dy/dx = y$ starting at $(0,1)$ and using two steps with $\Delta x = 0.05$ gives $y \approx 1.1025$ when $x = 0.1$.

Solution At $(0,1)$, the slope is 1 and $\Delta y = (1)(0.05) = 0.05$, so new $y = 1 + 0.05 = 1.05$. At $(0.05, 1.05)$, the slope is 1.05 and $\Delta y = (1.05)(0.05) = 0.0525$, so new $y = 1.05 + 0.0525 = 1.1025$ at $x = 0.1$.

In general, dy/dx may be a function of both x and y. Euler's method still works, as the next example shows.

Example 3 Approximate four points on the solution curve to $dy/dx = -x/y$ starting at $(0,1)$; use $\Delta x = 0.1$. Are the approximate values overestimates or underestimates?

Solution The results from Euler's method are in Table 11.3, along with the exact y-values (to two decimals) calculated from the equation of the circle $x^2 + y^2 = 1$, which is the solution curve through $(0,1)$. Since the curve is concave down, the approximate y-values are above the exact ones. (See Figure 11.26.)

Table 11.3 *Euler's method for $dy/dx = -x/y$, starting at $(0,1)$*

x	Approximate y-value	$\Delta y = (\text{Slope})\Delta x$	True y-value
0	1	$0 = (0)(0.1)$	1
0.1	1	$-0.01 = (-0.1/1)(0.1)$	0.99
0.2	0.99	$-0.02 = (-0.2/0.99)(0.1)$	0.98
0.3	0.97		0.95

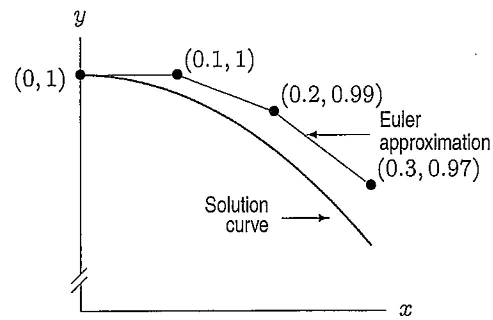

Figure 11.26: Euler's approximate solution to $dy/dx = -x/y$

The Accuracy of Euler's Method

To improve the accuracy of Euler's method, we choose Δx smaller. Let's go back to the differential equation $dy/dx = y$ and compare the exact and approximate values for different Δx's. The exact solution going through the point $(0,1)$ is $y = e^x$, so the exact values are calculated using this function. (See Figure 11.27.) Where $x = 0.1$,

$$\text{Exact } y\text{-value} = e^{0.1} \approx 1.1051709.$$

In Example 1 we had $\Delta x = 0.1$, and where $x = 0.1$,

$$\text{Approximate } y\text{-value} = 1.1, \quad \text{so the error} \approx 0.005.$$

In Example 2 we decreased Δx to 0.05. After two steps, $x = 0.1$, and we had

$$\text{Approximate } y\text{-value} = 1.1025, \quad \text{so error} \approx 0.00267.$$

Thus, it appears that halving the step size has approximately halved the error.

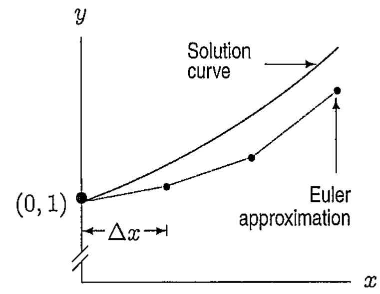

Figure 11.27: Euler's approximate solution to $dy/dx = y$

> The *error* in using Euler's method is the difference between the approximate value and the exact value. If the number of steps used is n, the error is approximately proportional to $1/n$.

Just as there are more accurate numerical integration methods than left and right Riemann sums, there are more accurate methods than Euler's for approximating solution curves. However, Euler's method will be all we need.

Exercises and Problems for Section 11.3

Exercises

1. Consider the differential equation $y' = x + y$ whose slope field is in Figure 11.17 on page 530. Use Euler's method with $\Delta x = 0.1$ to estimate y when $x = 0.4$ for the solution curves satisfying

(a) $y(0) = 1$ **(b)** $y(-1) = 0$.

2. **(a)** Use ten steps of Euler's method to determine an approximate solution for the differential equation $y' = x^3$, $y(0) = 0$, using a step size $\Delta x = 0.1$.
(b) What is the exact solution? Compare it to the computed approximation.
(c) Use a sketch of the slope field for this equation to explain the results of part (b).

3. Consider the solution of the differential equation $y' = y$ passing through $y(0) = 1$.

(a) Sketch the slope field for this differential equation, and sketch the solution passing through the point $(0, 1)$.
(b) Use Euler's method with step size $\Delta x = 0.1$ to estimate the solution at $x = 0.1, 0.2, \ldots, 1$.
(c) Plot the estimated solution on the slope field; compare the solution and the slope field.
(d) Check that $y = e^x$ is the solution of $y' = y$ with $y(0) = 1$.

4. **(a)** Use Euler's method to approximate the value of y at $x = 1$ on the solution curve to the differential equation

$$\frac{dy}{dx} = x^3 - y^3$$

that passes through $(0, 0)$. Use $\Delta x = 1/5$ (i.e., 5 steps).
(b) Using Figure 11.28, sketch the solution that passes through $(0, 0)$. Show the approximation you made in part (a).
(c) Using the the slope field, say whether your answer to part (a) is an overestimate or an underestimate.

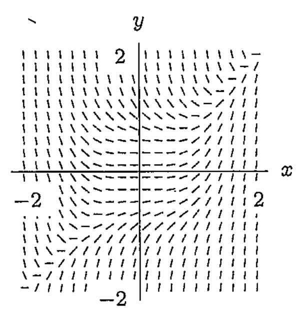

Figure 11.28: Slope field for $\frac{dy}{dx} = x^3 - y^3$