15. Pick out which functions are solutions to which differential equations. (Note: Functions may be solutions to more than one equation or to none; an equation may have more than one solution.)

(a) $\dfrac{dy}{dx} = -2y$	(I) $y = 2\sin x$
(b) $\dfrac{dy}{dx} = 2y$	(II) $y = \sin 2x$
(c) $\dfrac{d^2y}{dx^2} = 4y$	(III) $y = e^{2x}$
(d) $\dfrac{d^2y}{dx^2} = -4y$	(IV) $y = e^{-2x}$

16. Match the following differential equations and possible solutions. (Note the given functions may satisfy more than one equation or none, and some equations may have more than one solution.)

(a) $y'' = y$	(I) $y = \cos x$
(b) $y' = -y$	(II) $y = \cos(-x)$
(c) $y' = 1/y$	(III) $y = x^2$
(d) $y'' = -y$	(IV) $y = e^x + e^{-x}$
(e) $x^2y'' - 2y = 0$	(V) $y = \sqrt{2x}$

11.2 SLOPE FIELDS

In this section, we see how to visualize a first-order differential equation. We start with the equation

$$\frac{dy}{dx} = y.$$

Any solution to this differential equation has the property that the slope at any point is equal to the y coordinate at that point. (That's what the equation $dy/dx = y$ is telling us!) If the solution goes through the point $(0, 0.5)$, its slope there is 0.5; if it goes through a point with $y = 1.5$, its slope there is 1.5. See Figure 11.4.

In Figure 11.4 a small line segment is drawn at each of the marked points showing the slope of the curve there. Imagine drawing many of these line segments, but leaving out the curves; this gives the *slope field* for the equation $dy/dx = y$ in Figure 11.5. From this picture, we can see that above the x-axis, the slopes are all positive (because y is positive there), and they increase as we move upward (as y increases). Below the x-axis, the slopes are all negative, and get more so as we move downward. On any horizontal line (where y is constant) the slopes are constant.

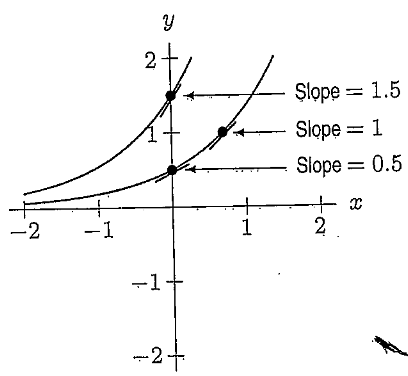

Figure 11.4: Solutions to $dy/dx = y$

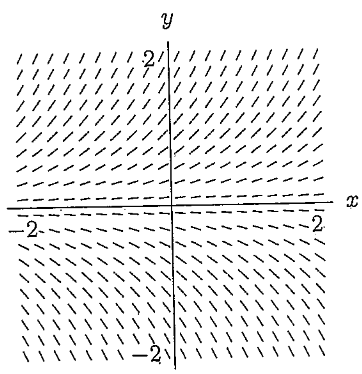

Figure 11.5: Slope field for $dy/dx = y$

In the slope field we can see the ghost of the solution curve lurking. Start anywhere on the plane and move so that the slope lines are tangent to our path; we trace out one of the solution curves. We think of the slope field as a set of signposts pointing in the direction we should go at each point. In this case, the slope field should trace out exponential curves of the form $y = Ce^x$, the solutions to the differential equation $dy/dx = y$.

Example 1 Figure 11.6 shows the slope field of the differential equation $dy/dx = 2x$.

(a) How does the slope field vary as we move around the xy-plane?

(b) Compare the solution curves sketched on the slope field with the formula for the solutions.

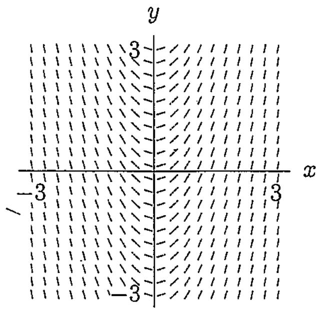

Figure 11.6: Slope field for $dy/dx = 2x$

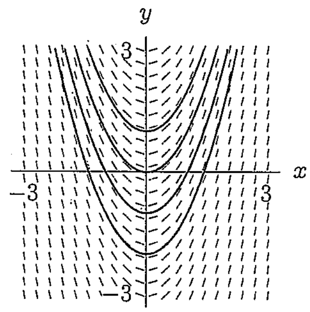

Figure 11.7: Some solutions to $dy/dx = 2x$

Solution

(a) In Figure 11.6 we notice that on a vertical line (where x is constant) the slopes are constant. This is because in this differential equation dy/dx depends on x only. (In the previous example, $dy/dx = y$, the slopes depended on y only.)

(b) The solution curves in Figure 11.7 look like parabolas. By antidifferentiation, we see that the solution to the differential equation $dy/dx = 2x$ is

$$y = \int 2x\,dx = x^2 + C,$$

so the solution curves really are parabolas.

Example 2 Using the slope field, guess the form of the solution curves of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Solution The slope field is shown in Figure 11.8. On the y-axis, where x is 0, the slope is 0. On the x-axis, where y is 0, the line segments are vertical and the slope is infinite. At the origin the slope is undefined, and there is no line segment.

The slope field suggests that the solution curves are circles centered at the origin. Later we see how to obtain the solution analytically, but even without this, we can check that the circle is a solution. We take the circle of radius r,

$$x^2 + y^2 = r^2,$$

and differentiate implicitly, thinking of y as a function of x. Using the chain rule, we get

$$2x + 2y \cdot \frac{dy}{dx} = 0.$$

Solving for dy/dx gives our differential equation,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

This tells us that $x^2 + y^2 = r^2$ is a solution to the differential equation.

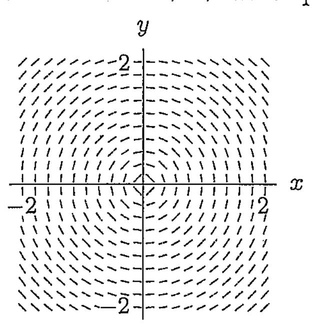

Figure 11.8: Slope field for $dy/dx = -x/y$

The previous example shows that the solution to a differential equation may be an implicit function.

Example 3 The slope fields of $dy/dt = 2 - y$ and $dy/dt = t/y$ are in Figures 11.9 and 11.10.

(a) On each slope field, sketch solution curves with initial conditions

(i) $y = 1$ when $t = 0$ (ii) $y = 0$ when $t = 1$ (iii) $y = 3$ when $t = 0$

(b) For each solution curve, what can you say about the long-run behavior of y? For example, does $\lim_{t\to\infty} y$ exist? If so, what is its value?

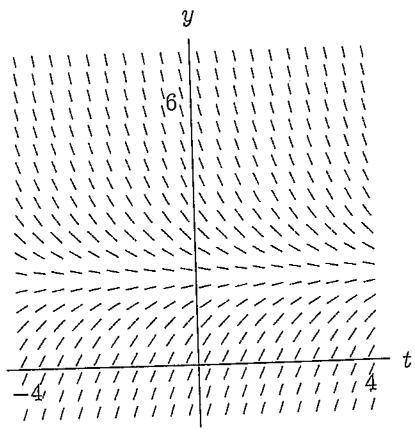

Figure 11.9: Slope field for $dy/dt = 2 - y$

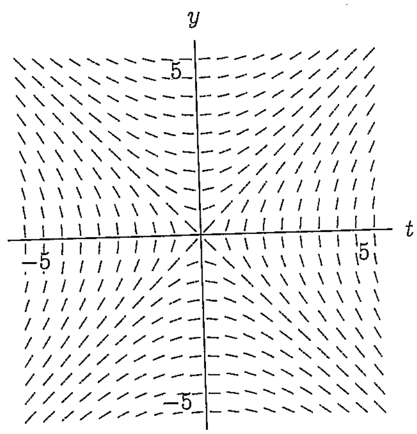

Figure 11.10: Slope field for $dy/dt = t/y$

Solution (a) See Figures 11.11 and 11.12.

(b) For $dy/dt = 2 - y$, all solution curves have $y = 2$ as a horizontal asymptote, so $\lim_{t\to\infty} y = 2$. For $dy/dt = t/y$, as $t \to \infty$, it appears that either $y \to t$ or $y \to -t$.

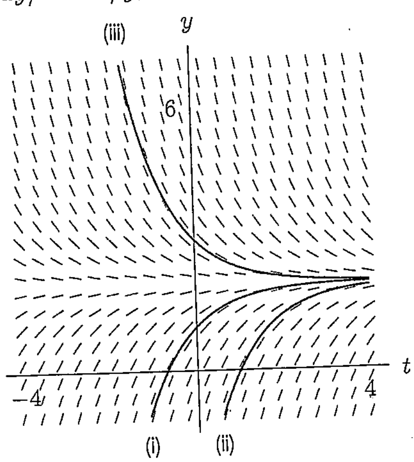

Figure 11.11: Solution curves for $dy/dt = 2 - y$

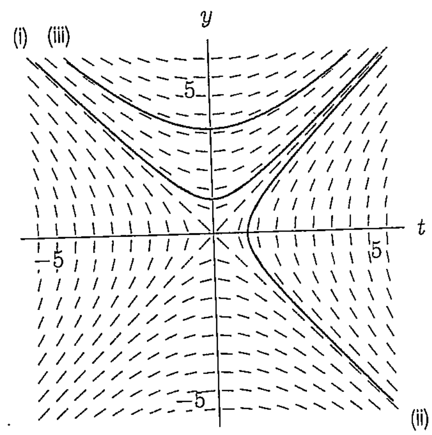

Figure 11.12: Solution curves for $dy/dt = t/y$

Existence and Uniqueness of Solutions

Since differential equations are used to model many real situations, the question of whether a solution is unique can have great practical importance. If we know how the velocity of a satellite is changing, can we know its velocity at any future time? If we know the initial population of a city, and we know how the population is changing, can we predict the population in the future? Common sense says yes: if we know the initial value of some quantity and we know exactly how it is changing, we should be able to figure out its future value.

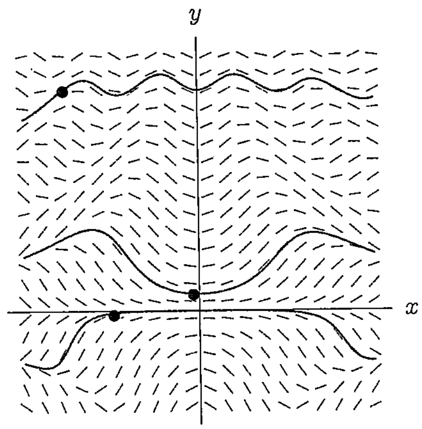

Figure 11.13: There's one and only one solution curve through each point in the plane for this slope field (Dots represent initial conditions)

In the language of differential equations, an initial value problem (that is, a differential equation and an initial condition) almost always has a unique solution. One way to see this is by looking at the slope field. Imagine starting at the point representing the initial condition. Through that point there is usually a line segment pointing in the direction of the solution curve. By following these line segments, we trace out the solution curve. See Figure 11.13. In general, at each point there is one line segment and therefore only one direction for the solution curve to go. The solution curve *exists* and is *unique* provided we are given an initial point. Notice that even though we can draw the solution curves, we may have no simple formula for them.

It can be shown that if the slope field is continuous as we move from point to point in the plane, we can be sure that a solution curve exists everywhere. Ensuring that each point has only one solution curve through it requires a slightly stronger condition.

Exercises and Problems for Section 11.2

Exercises

1. Sketch three solution curves for each of the slope fields in Figures 11.14 and 11.15.

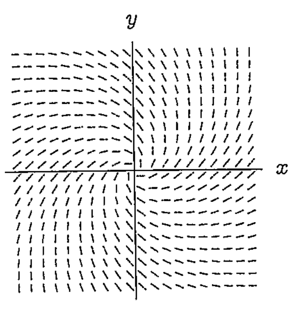

Figure 11.14

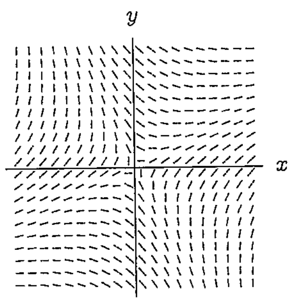

Figure 11.15

2. The slope field for the equation $y' = x(y-1)$ is shown in Figure 11.16.

(a) Sketch the solutions passing through the points
(i) $(0,1)$ (ii) $(0,-1)$ (iii) $(0,0)$

(b) From your sketch, write down the equation of the solution with $y(0) = 1$.

(c) Check your solution to part (b) by substituting it into the differential equation.

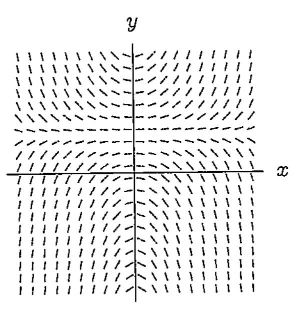

Figure 11.16