Implicit Differentiation

Quite often one encounters an equation relating two variables where it is impossible or inconvenient to solve for one of them in terms of the other. An example is

$$y^3 - xy = -6.$$

As x varies in this equation, y must also vary in order to maintain equality. The points (x, y) satisfying this equation lie along some curve in the (x, y) plane.

It is inconvenient to solve this equation for y in terms of x. The Chain Rule can nonetheless be used to determine the derivative of $y(x)$, provided we know that this derivative exists. Given a formula for $y'(x)$ and one ordered pair (x_0, y_0) satisfying the equation, the Microscope Approximation (or, equivalently, the tangent line) can be used to approximate values of $y(x)$ for x near x_0.

It is common to write y rather than the more complete $y(x)$. For this reason, it is often very helpful to use the notation $\frac{d}{dx}$ to indicate the variable with respect to which one is differentiating. On the other hand, if you are clear about which variable this is, you can also use the "prime" notation. The strategy is:

i) Differentiate the full equation with respect to the independent variable. You will need to use the Chain Rule, among other rules.

ii) Solve for the derivative.

Example: (both notations are shown)

$$y^3 - xy = -6$$

$$3y^2\, y' - (1 \cdot y + x \cdot y') = 0$$

$$y' = \frac{y}{3y^2 - x}$$

or

$$\frac{d}{dx}y^3 - \frac{d}{dx}xy = \frac{d}{dx}(-6)$$

$$3y^2\frac{dy}{dx} - \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = \frac{y}{3y^2 - x}$$

1. Assuming this derivative exists, use implicit differentiation to find the derivative of y with respect to x if
$$x^2 - xy + y^2 = 7.$$

2. Confirm that the point $(x, y) = (-1, 2)$ satisfies the equation $x^2 - xy + y^2 = 7$.

3. Based on your work so far, if $x^2 - xy + y^2 = 7$, what are the following values?

$$y(-1) = \qquad\qquad y'(-1) =$$

4. Let C denote the curve whose points (x, y) satisfy the equation $x^2 - xy + y^2 = 7$. Find the equation for the tangent line to C at the point $(-1, 2)$.

5. Use this tangent line to estimate $y(-.9)$.

An Extra:

Are there any points (x, y) on the curve C where the tangent line is *vertical*? If so, find them.