The Product and Quotient Rules

Warm-Up

Recall that by Taylor's Theorem, if f is differentiable at a, then

$$f(x) = f(a) + f'(a)(x-a) + E(x).$$

1. What are the values of the following limits?

$$\lim_{x\to a} E(x) = \qquad\qquad \lim_{x\to a} \frac{E(x)}{x-a} =$$

THEOREM: (Product Rule) If f and g are both differentiable at a, then their product $f \cdot g$ is also differentiable at a and

$$(f\cdot g)'(a) = f'(a)\cdot g(a) + f(a)\cdot g'(a).$$

Proof: By Taylor's Theorem,

$$f(x) = f(a) + f'(a)(x-a) + E_f(x),$$
$$g(x) = g(a) + g'(a)(x-a) + E_g(x).$$

Therefore,

$$\begin{aligned}
&(f\cdot g)'(a)\\
&= \lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x-a}\\
&= \lim_{x\to a} \frac{\big(f(a) + f'(a)(x-a) + E_f(x)\big)\cdot \big(\big(g(a) + g'(a)(x-a) + E_g(x)\big) - f(a)g(a)}{x-a}\\
&= \lim_{x\to a} \frac{f(a)g(a) + f'(a)g(a)(x-a) + f(a)g'(a)(x-a) - f(a)g(a)}{x-a}\\
&\quad + \lim_{x\to a} \frac{f(a)E_g(x) + g(a)E_f(x) + E_f(x)E_g(x)}{x-a}\\
&= \lim_{x\to a}\left(f'(a)g(a) + f(a)g'(a) + f(a)\frac{E_g(x)}{x-a} + g(a)\frac{E_f(x)}{x-a} + E_f(x)\frac{E_g(x)}{x-a}\right)\\
&= \lim_{x\to a} f'(a)g(a) + \lim_{x\to a} f(a)g'(a)\\
&\quad + f(a)\cdot \lim_{x\to a}\frac{E_g(x)}{x-a} + g(a)\cdot\lim_{x\to a}\frac{E_f(x)}{x-a} + E_f(x)\cdot\lim_{x\to a}\frac{E_g(x)}{x-a}\\
&= f'(a)g(a) + f(a)g'(a).
\end{aligned}$$

2. Give reasons justifying each step of this proof.

COROLLARY: (Quotient Rule) If f and g are differentiable at a, and if $g(a) \neq 0$, then their quotient f/g is also differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}, \quad g(a) \neq 0.$$

Proof: Let $Q(x) = f(x)/g(x)$. In this proof we will assume that the quotient $Q(x)$ is differentiable at a. (We can confirm this next week once we have the Chain Rule.) Then

$$\begin{aligned}
Q(x)g(x) &= f(x) \\
Q'(a)g(a) + Q(a)g'(a) &= f'(a) \\
Q'(a)g(a) &= f'(a) - Q(a)g'(a) \\
Q'(a) &= \frac{f'(a) - Q(a)g'(a)}{g(a)} \\
&= \left(f'(a) - \frac{f(a)}{g(a)} \cdot g'(a)\right)\Big/ g(a) \\
&= \left(\frac{f'(a)g(a) - f(a)g(a)}{g(a)}\right)\Big/ g(a) \\
&= \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.
\end{aligned}$$

2. Give reasons justifying each step of this proof.

Applications of the Product and Quotient Rules

The Power Rule

$$\begin{aligned}
f(x) = x &\implies f'(x) = 1 \\
f(x) = x^2 = x \cdot x &\implies f'(x) = 1 \cdot x + x \cdot 1 = 2x \\
f(x) = x^3 = x^2 \cdot x &\implies f'(x) = 2x \cdot x + x^2 \cdot 1 = 3x^2 \\
\vdots \qquad\qquad &\vdots \\
f(x) = x^n \qquad &\implies \qquad\qquad f'(x) = \\
f(x) = x^{n+1} = x^n \cdot x &\implies f'(x) = \\
\vdots \qquad\qquad &\vdots
\end{aligned}$$

Examples

3. Complete the following statements:

   If $f(x) = 3x^2 - 4x + 2$, then $f'(x) =$

   If f is a polynomial of degree n, then $f'(x)$ is a polynomial of degree ....

The Derivative of the Sine Function

We will assume that the sine function is differentiable. (We can confirm this next week once we have the Chain Rule.) Our strategy will be to start with the trigonometric identity

$$\cos^2(x) + \sin^2(x) = 1,$$

then apply the Product Rule and solve for $\sin'(x)$. Recall that $\cos'(x) = -\sin(x)$.

$$\begin{aligned}
\cos(x)\cdot\cos(x) + \sin(x)\cdot\sin(x) &= 1 \\
\cos'(x)\cos(x) + \cos(x)\cos'(x) + \sin'(x)\sin(x) + \sin(x)\sin'(x) &= 0 \\
-\sin(x)\cdot\cos(x) + \cos(x)\cdot\big(-\sin(x)\big) + \sin'(x)\sin(x) + \sin(x)\sin'(x) &= 0 \\
-2\sin(x)\cos(x) + 2\sin'(x)\sin(x) &= 0 \\
2\sin'(x)\sin(x) &= 2\sin(x)\cos(x) \\
\text{If } \sin(x) \neq 0, \text{ then} \qquad \sin'(x) &= \cos(x).
\end{aligned}$$

The one difficulty with this derivation is that it does not apply when $\sin(x) = 0$. Do you see why? As homework you will find the derivative of the sine function directly from the definition of the derivative. This way you will see that the formula $\sin'(x) = \cos(x)$ holds even when $\sin(x) = 0$.

The Derivative of the Tangent Function

The tangent function is defined as the quotient of the sine function by the cosine function:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \qquad \cos(x) \neq 0.$$

Applying the quotient rule,

$$\begin{aligned}
\tan'(x) &= \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{\cos^2(x)} \\
&= \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} = \sec^2(x).
\end{aligned}$$