

Introduction

In Class 14 you were introduced to the notion of a **limit** in defining the instantaneous velocity of a moving object. This lab has three goals:

- A. give you a more precise understanding of what is meant by a limit;
- B. use the language of limits to express the fact that the graphs of some functions have no breaks in them;
- C. use the idea of a limit to precisely define (when this is possible) the slope of the graph of a function, even if that function is not linear.

In the first part of this lab you will use Excel. In the second part of the lab you will be introduced to another software tool, *Derive*. In the final part of the lab you will return to Excel.

Limits

We'll begin with an example. But first, take a minute to remind yourself about the sine function by evaluating the following:

$$\sin(-\pi) = \quad , \quad \sin(-\pi/4) = \quad , \quad \sin(0) = \quad , \quad \sin(\pi/4) = \quad , \quad \sin(\pi) = \quad .$$

You may also wish to graph the sine function on your graphing calculator. (By the way, although you can approximately evaluate these expressions using your calculator, you should know the exact values by heart.)

Now on to the example. Let $p(x) = \sin(x)/x$.

1. What is the *natural domain* for p ? That is, can x be any real number or are there values that x cannot have?

We are now going to investigate $p(x)$ for values of x near 0. However, rather than just picking a bunch of values for x and seeing what $p(x)$ equals in each case, we are going to be more systematic. Specifically, we are first going to evaluate $p(x)$ for sequences of positive values of x that get closer and closer to 0. Then we are going to evaluate $p(x)$ for sequences of negative values of x that get closer and closer to 0.

With your lab group, go now to a computer, start *Excel*, and open the Wk06F00Lab workbook in the Shared . . . /Math Courses/Math 110 directory. In the upper left hand corner of the sheet labeled p(x) you will see column headings for x and $p(x) = \sin(x)/x$. Using your cursor, click on each cell and make sure you understand the formulas associate with each.

The Limit of $p(x)$ as $x \rightarrow 0^+$.

Copy the formulas for the last row down the columns. Note that all the x values are positive.

2. What value is x approaching? (See if you can also convince yourself of this by looking at the formula producing this sequence of x values.)
3. As x approaches this value, what value does $p(x)$ seem to approach?

Use the Chart Wizard to create a scatter plot of $p(x)$ vs. x and confirm visually what you have already observed in table.

You should have observed that as x approaches 0 through positive values, $p(x)$ approaches 1. To see this even more clearly, go to the top of the column in the spreadsheet next to $p(x)$. Label this column $p(x) - 1$, and in the next cell in that column enter the formula = B2 - 1. Copy this formula down to the rest of the cells in that column to fill out the table. Similarly, label the *next* column as $|p(x) - 1|$, then enter and copy the formula needed to compute the absolute value of $p(x) - 1$.

4. Based on your table, find values of x to complete the following statements. (You may need to extend the table to answer some of these questions.)

If $0 < x < \quad$, then $0 < |p(x) - 1| < 0.1$.

If $0 < x < \quad$, then $0 < |p(x) - 1| < 0.01$.

If $0 < x < \quad$, then $0 < |p(x) - 1| < 0.001$.

If $0 < x < \quad$, then $0 < |p(x) - 1| < 0.0001$.

5. Based on this experience, do you believe that it is always possible, in principle, to complete a statement of the form:

“If $0 < x < \quad$, then $0 < |p(x) - 1| < \text{tolerance}$,”

no matter how small a positive value we choose for the “tolerance” ?

The formal, symbolic way of stating that this is always possible is:

$$\lim_{x \rightarrow 0^+} p(x) = 1,$$

read, “As x approaches 0 from ABOVE, $p(x)$ approaches 1,” or alternatively, “The limit of $p(x)$, as x approaches 0 from above, is 1.” Note some special symbols: the arrow “ \rightarrow ” stands for “approaches” and the superscript “+” on the 0 stands for “from above.”

The Limit of $p(x)$ as $x \rightarrow 0^-$.

6. By changing the value in ONE cell in the spreadsheet you can study the behavior of $p(x)$ as x approaches 0 through NEGATIVE values. Make this change, then complete the following statements:

If $\quad \quad \quad < x < 0$, then $0 < |p(x) - 1| < 0.1$.

If $\quad \quad \quad < x < 0$, then $0 < |p(x) - 1| < 0.01$.

If $\quad \quad \quad < x < 0$, then $0 < |p(x) - 1| < 0.001$.

If $\quad \quad \quad < x < 0$, then $0 < |p(x) - 1| < 0.0001$.

7. Based on this experience, do you believe that it is always possible, in principle, to complete a statement of the form:

“If $\quad \quad \quad < x < 0$, then $0 < |p(x) - 1| < \text{tolerance}$,”

no matter how small a positive value we choose for the “tolerance” ?

The formal, symbolic way of stating that this is always possible is:

$$\lim_{x \rightarrow 0^-} p(x) = 1,$$

read, “As x approaches 0 from BELOW, $p(x)$ approaches 1,” or alternatively, “The limit of $p(x)$, as x approaches 0 from below, is 1.” Note another special symbol: the superscript “-” on the 0 stands for “from below.”

The Limit of $p(x)$ as $x \rightarrow 0$.

The following three statements are true for our example:

a) $\lim_{x \rightarrow 0^-} p(x)$ exists;

b) $\lim_{x \rightarrow 0^+} p(x)$ exists;

a) $\lim_{x \rightarrow 0^-} p(x) = 1 = \lim_{x \rightarrow 0^+} p(x)$.

The formal, symbolic way of summarizing these three statements is:

$$\lim_{x \rightarrow 0} p(x) = 1,$$

read, “As x approaches 0, $p(x)$ approaches 1,” or alternatively, “The limit of $p(x)$, as x approaches 0, is 1.”

Remarks on Limits

In this second unit of the course, many of the statements we make will involve limits. You probably already have a good intuition about what a limit is, and now you have seen – in the context on one example – a very careful explanation of exactly what mathematicians mean by a limit.

Yet even in this example we were not able to actually PROVE that $\lim_{x \rightarrow 0} p(x) = 1$. Such a proof is possible with more advanced mathematics. In this course, however, we will continue to rely on your intuition concerning limits while gradually giving you more and more tools to reason with them.

Continuous Functions

Intuitively, a function f is continuous if its graph has no breaks. A break in a graph could occur for two sorts of reasons. It might be that the function is not defined at the point where the break occurs. It may also be that the function value takes a sudden jump there. If a function's graph has a break at a given input value a , then the function is said to be *discontinuous* at a . If the function is not discontinuous at a point, then it is said to be *continuous* at that point.

An example will help make this more clear and will also lead us to a formal definition of continuity in terms of limits.

8. You'll need to use your "mind's eye," not your graphing calculator to answer this question! "Is $p(x) = \sin(x)/x$ continuous at $x = 0$? Explain.

9. Define a new function q as

$$q(x) = \begin{cases} \sin(x)/x, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Is q continuous at $x = 1$? Explain.

Examples like this one have led mathematicians to define continuity in terms of limits, as follows:

Definition: (Continuity) A function f is said to be **continuous at the point** a if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

- Using this definition, explain why $p(x)$ is NOT continuous at 0 but $q(x)$ IS continuous at 0.

Local Linearity

We are now going to study the graph of $q(x)$ closely in a neighborhood of 0. It is possible to use your graphing calculator to do this. However, we will introduce you now to another powerful software tool available in the computer labs: *Derive*. To begin, *minimize* your *Excel* spreadsheet, then click on the **Mathematics** icon, then on the **Derive** icon.

On the tool bar at the top of the window, click on **Author** then on **Expression**. In the box which then opens up, type $\sin(x)/x$, then click on **Simplify**. This expression will now appear in the main window.

Place your cursor over the second icon from the right on the tool bar and confirm that this selects the “2-D Plot.” Then click on this icon. A window now opens up with coordinate axes. At the top of the tool bar, click on **Plot!** and the graph of the function is plotted.

(Note: although you entered the expression $\sin(x)/x$, we can regard the graph we see as the graph of $q(x)$ rather than $p(x)$. This is because we could never actually see a hole in the graph where the point $(0, 1)$ is missing from the graph of $p(x)$.)

- Either sketch or briefly describe the main features of this graph.

On the graph you will notice a crosshairs. Click on the point $(0, 1)$ to move the crosshairs to this point. Place your cursor over the sixth icon from the left on the tool bar and confirm that this selects “Center on cross.” Then click on that icon.

Now place your cursor over the fourth icon from the right on the tool bar and confirm that it selects “Zoom both in.” Then click on that icon and notice how the graph changes. This action corresponds to looking at the graph around the crosshairs as if you were looking through a microscope.

12. Repeat the process of “zooming in” over and over. (You may occasionally need to also move the crosshairs to the point $(0, 1)$ and recenter the graph on the crosshairs.) How does the portion of the graph you are looking at change?

If the graph of a function f appears more and more like a line as you zoom in closer and closer to a point $(a, f(a))$, then f is said to be **locally linear** at a . From your experience with lines, it will come as no surprise that we will be able to define the slope or *instantaneous rate of change* of a function at a point where it is locally linear.

13. What do you think that the numerical value of the instantaneous rate of change of $q(x)$ is at the point $x = 0$? Why? Why does it not make sense to talk about the instantaneous rate of change of $p(x)$ at 0 ?

The Instantaneous Rate of Change of q at 0

In *Derive*, select **File Exit** from the tool bar and choose NO when prompted to save changes. Then maximize your *Excel* window and select the second worksheet, labeled $q'(0)$.

14. Use this worksheet to decide whether or not the following limits exist, and if they exist, what their values are. (You do not need to add additional columns this time.)

$$\lim_{\Delta x \rightarrow 0^-} \frac{q(0 + \Delta x) - q(0)}{\Delta x} =$$

$$\lim_{\Delta x \rightarrow 0^+} \frac{q(0 + \Delta x) - q(0)}{\Delta x} =$$

15. Based on these results, does the following limit exist? If so, what is its value?

$$\lim_{\Delta x \rightarrow 0} \frac{q(0 + \Delta x) - q(0)}{\Delta x} =$$

15. Why is this a sensible way to define $q'(0)$, the instantaneous rate of change of q at 0?

Preparing Your Lab Report

As before, your report should consist of a cover page with the title of the report and the names and signatures of your lab group members. **Please also write your lab section day and time.** Each person (in a group of three) should draft one of the three parts, and the group should meet *before the next lab meeting* to read and comment on these drafts before submitting the final report. Smaller groups should divide the work equally among themselves.

Part 1

Read very carefully the definitions for the limit of $p(x)$ as $x \rightarrow 0^-$, as $x \rightarrow 0^+$ and as $x \rightarrow 0$. How would you modify these definitions to define the limit of some other function f as $x \rightarrow a^-$, as $x \rightarrow a^+$ and as $x \rightarrow a$, where a is some specific value? Stay as close to the original definitions as possible, changing only what is absolutely necessary.

Part 2

Carefully explain why $p(x)$ is not continuous at 0 but $q(x)$ is continuous there. Refer to the definition of continuity given in this lab as well as the numerical evidence you collected and your graphical intuition.

Part 3

Why is the definition of $q'(0)$ given in this lab reasonable? Discuss the graphical and numerical evidence you have that $q'(0) = 0$.