Introduction

The goal of today's lab is explore the S-I-R Model of Disease by using Microsoft Excel as a tool which can rapidly implement Euler's Method to find approximate solutions to various initial value problems.

§1. The SIR model and the threshold value

$$\begin{aligned} S'(t) &= -aS(t)I(t) + cR(t), \qquad S(t_0) = S_0 \\ I'(t) &= aS(t)I(t) - bI(t), \qquad I(t_0) = I_0 I \\ R'(t) &= bI(t) - cR(t), \qquad R(t_0) = R_0 \end{aligned}$$

1. When an illness passes through a population, it is said to be an *epidemic* as long as the size of the infected population is increasing, that is to say, when more people are getting sick each day than are recovering. Once the number of recoveries each day exceeds the number of new cases of the illness, the disease is no longer considered an epidemic. The illness has *peaked* when the number of infected persons is greatest. Before its peak, a disease is considered an epidemic; after the peak, it is no longer thought of as an epidemic. It is possible for a disease to avoid detection until after its peak has passed.

   (a) Go to `H:\ Math Courses \ Math110 \ Wk05F00Lab.xls` and open the file. Open the Euler's Method table on the sheet labeled `SIR Model and Euler's Table`. You will find an initial value problem which represents our basic SIR Model. Make the table large enough so that it covers the first 100 days of the epidemic. The value for Δt is given as 1. Once you have produced the Euler's table, check out the graph on the sheet `Time versus S, I and R`.

   (b) Whether a disease has an epidemic stage or not depends on a number of things, including how infectious the disease is and how quickly an infected person recovers. In fact, we showed in class that whether a disease has an epidemic stage is completely dependent on whether the number of susceptibles is above a particular *threshold value*. This threshold value is given by the ratio of the two parameters a and b in the standard SIR model.

   $$S_{threshold} = b/a$$

   What is the threshold value for the model used in the spreadsheet?

   (c) Observe how the graphs change when you change the value of S_0 from 45400 to 15000 to 7500 and then to 5000. What do the graphs look like when S_0 is exactly equal to $S_{threshold}$? For which values of S_0 does one see a "peak" in the infecteds?

(d) Return S_0 to 45400. Suppose a quarantine is implemented which results in the transmission coefficient being reduced by a factor of 10, from the original value. What change do you expect in the graphs of $S(t), I(t), R(t)$ versus t? Implement the change in the model solved in the spreadsheet. Do the computed results match your expectations?

(e) Suppose the Backstreet Boys, Britney Spears, N*SYNC, Macy Gray *and* Limp Bizkit give a joint concert so that the transmission coefficient increases by a factor of 20. What happens to the solution graphs? Do you *believe* these results? How would you improve them? (You probably should change the table to just compute 50 days of the epidemic instead of 100—don't forget to change the Source Data in the Data Range Tab on the graph also.)

(f) Thanks to medical advances in recent years, there are two new drugs on the market, one of which can reduce the duration of the disease from 14 days to 7 days, and the other (much more expensive) can reduce the duration of the disease to a mere 24 hours. How do the solution curves change in each case?

(g) Notice that the S-I-R model as written in this lab has a third parameter, c, known as the *immunity loss parameter.* This represents what would happen if people who recover from the disease lost their immunity from re-infection after $1/c$ days. How do you think this affects the solution generated by Excel? Run the spreadsheet with 5 different values of c, from $c = .01, .1, 1$ and 10. (Note what happens to the Recovered population over time. Does this make sense? What do you have to do to "fix" the situation?)

§2. **Proportionality of Pointwise Euler's Error to Stepsize**

In this section of the lab we want to investigate how the error in Euler's Method varies with the size of Δt. We will do so by calculating the number of susceptibles after 20 days, $S(20)$ and then changing the sign of Δt and computing the value of $S(0)$. So, we will go forward in time $20/\Delta t$ steps, and then back again the same number of steps.

2. We will do our calculation with $a = .00001, 1/b = 14, c = 0, S_0 = 45400, I_0 = 4200, R_0 = 400$

   Complete the following table

Δt	$S(20)$	$S(0)$	$S(0) - S_0$	$\dfrac{S(0) - S_0}{\Delta t}$
0.01				
0.1				
0.25				
0.5				
1.0				
2.0				
4.0				

In the table $S(20)$ and $S(0)$ are the computed values of the susceptibles after $t = 0$ and $t = 20$, respectively. In order to obtain $S(20)$ go to the **SIR Model** spreadsheet, type in the first value of $\Delta t = 0.01$ and copy the cells far enough until you can read off a value for $S(20)$ which you can write in the blank space in the table on this page.

In order to find an estimate for $S(0)$ you have to change Δt to be -0.01, and copy cells until you are able to read off a value for $S(0)$. (Be careful! On which line of the spreadsheet should your first negative value of $\Delta t = -0.01$ appear? The $t = 20.00$ line? The $t = 20.01$ line? The $t = 19.99$ line?)

You then repeat this procedure for the given values of Δt in the table on this page, and you can select two other values of Δt to try out also. When you change your value of Δt also take a look at the graph of the solutions. To change the Source Data, just right-click on the graph itself, and then select exactly which cells you want to be graphed. (You will have to update this if you want to be consistently looking at 20 days of data).

To complete the rest of the table, especially the $S(0) - S_0$ and $\frac{S(0)-S_0}{\Delta t}$ calculations, you may find it easy to have Excel calculate them. Note that $S(0) - S_0$ is the what we call the *pointwise*

Euler's Method error, or the error in approximation to $S(t)$ at a particular point, and we are trying to show that this error is directly proportional to Δt.

Preparing Your Lab Report

Your report should consist of a cover page with the title of the report and the names and signatures of your lab group members. **Also indicate your Lab Section, Lab Time and Instructor**, e.g. Section 2: Thursday 8:30 am (Buckmire). Each person (in a group of three) should complete a first draft of one of the three parts, and the group should meet to read and discuss these drafts before submitting the final report. The final report is DUE IN NEXT WEEK'S LAB. Grading will be Credit/No Credit with, however, a high standard for receiving credit. If you do not initially receive credit you will have one week to revise the report to correct any problems with it. If you have questions whether your draft will likely receive Credit, you should visit your instructor in office hours and have your draft lab report evaluated.

You will need to refer to your Excel worksheets for this lab to answer these questions.

Part 1

State the initial value problem you worked on in this lab. Explain the significance and effect of the parameters on the model. When did you see a "peak" in the $I(t)$ graph? When did you see a peak in the $R(t)$ graph? Explain why these peaks occur and what they mean in terms of the model of the disease. Which parameters in the initial value problem does the solution seem the most sensitive to? (In other words, relatively small changes produce significant changes in the appearance of the solution curves).

Part 2

As you changed your Δt you produced successive approximations of the solution curves $S(t)$, $I(t)$ and $R(t)$. What values of Δt produced "smooth" graphs? Which values produced "choppy" results? What other information from the graphs leads you to believe that using Euler's Method with smaller values of Δt is preferred?

Part 3

Produce a graph which illustrates the direct proportionality relationship between the error Euler's Method makes in approximating the solution function $S(t)$ and the value of Δt used in each calculation. Are you able to determine the exact value of the constant of proportionality? Why or why not?