Recalling Newton's Law of Cooling

In Lab 2 you studied a model developed to predict the temperature of hot water in a covered aluminum cup, partially submerged in an ice bath in a covered cooler. Under the assumption that the ambient temperature A remained constant, the model (Newton's Law of Cooling) took the form of a rate equation

$$H'(t) = -k \cdot (H(t) - A)$$

together with an intial condition. The values of the constants $k > 0$ and A were estimated from the data.

Refining the Model

In Class 8, this model was refined to take into account the fact that the ambient environment consisted of two parts – an ice bath and the air above the ice bath and the covered aluminum cup containing the hot water. Because some of the aluminum cup was exposed above the surface of the ice bath, heat from the water could be conducted through the aluminum and radiate into the air. This would raise the temperature of the air (violating our previous assumption that the ambient temperature remained constant).

Later, after the water had been cooled quite a bit, heat from the air could be conducted through the aluminum back into the water. This could explain why the water eventually cooled more slowly than predicted by the simple Newton's Law of Cooling model we looked at in Lab 2. These considerations lead to the following model:

Let ...

$H(t)$ denote the temperature of the water, in degree Celsius

$A(t)$ denote the temperature of the ambient air, in degree Celsius

$H'(t)$ denote the rate of change of $H(t)$, in degrees Celsius per second

$H'_A(t)$ denote the rate of change of $H(t)$ due to interaction with the ambient air

$H'_B(t)$ denote the rate of change of $H(t)$ due to interaction with the ice bath

$A'(t)$ denote the rate of change of $A(t)$, in degrees Celsius per second

$A'_H(t)$ denote the rate of change of $A(t)$ due to interaction with the water

$A'_B(t)$ denote the rate of change of $A(t)$ due to interaction with the ice bath

Note that the temperature of the ice bath remains a constant 0° C. Then

$$H'(t) = H'_A(t) + H'_B(t)$$
$$A'(t) = A'_H(t) + A'_B(t)$$

To model each of the rates $H'_A(t)$, $H'_B(t)$, $A'_H(t)$ and $A'_B(t)$ we can use the same ideas that led us to Newton's Law of Cooling in the first place.

1. You should be able to explain each of these:

$$H'_A(t) = -k_1 \cdot (H(t) - A(t)), \qquad H'_B(t) = -k_2 \cdot (H(t) - 0)$$
$$A'_H(t) = -k_3 \cdot (A(t) - H(t)), \qquad A'_B(t) = -k_4 \cdot (A(t) - 0)$$

   where k_1, k_2, k_3 and k_4 are positive constants.

Putting these all together we have the model:

$$H'(t) = -k_1 \cdot (H(t) - A(t)) - k_2 \cdot (H(t) - 0), \qquad H(0) = 84.98$$
$$A'(t) = -k_3 \cdot (A(t) - H(t)) - k_4 \cdot (A(t) - 0), \qquad A(0) = A_0$$

Estimating Parameters

As it stands, there are *five* parameters in this model that are not yet determined: k_1, k_2, k_3, k_4 and the initial value A_0 of the ambient air temperature. From the experimental data, we also have the following observations:

a) $H'(0) \approx -0.7$ (estimates of this initial rate vary according to the time interval used);

b) after 45 minutes, the water temperature is a little above 3.5° C and changing very slowly.

The second observation supports – in the absence of actual measurements – setting

$$A_0 = 3.5.$$

The first observation is a constraint on the values of k_1 and k_2,

$$\begin{aligned} -0.7 &\approx -k_1 \cdot (H(0) - A(0)) - k_2 \cdot (H(0) - 0) \\ &= -k_1 \cdot (84.98 - 3.5) - k_2 \cdot (84.98 - 0) \\ &= -81.48 \cdot k_1 - 84.98 \cdot k_2, \end{aligned}$$

but tells us nothing about k_3 and k_4.

Were data available for $A(t)$, a variety of techniques could be used to estimate all these parameters. In the absence of this data, trial and error has been used to find parameter values which yield a solution $H(t)$ matching the data quite a bit better than that of the simple Newton's Law of Cooling model. The full revised model, with these parameter values, is:

$$H'(t) = -0.0065 \cdot (H(t) - A(t)) - 0.00075 \cdot (H(t) - 0), \qquad H(0) = 84.98$$
$$A'(t) = -0.0013 \cdot (A(t) - H(t)) - 0.0005 \cdot (A(t) - 0), \qquad A(0) = 3.5$$

Approximating Solutions Using Euler's Method

Even though this model has *two* rather than one rate equations, you can still use Euler's Method to produce a piecewise linear approximation to $H(t)$, and at the same time a piecewise linear approximation to $A(t)$:

$$\begin{aligned} \Delta H &\approx H'(t)\Delta t \\ \Delta A &\approx A'(t)\Delta t \\ H(t + \Delta t) &= H(t) + \Delta H \\ A(t + \Delta t) &= A(t) + \Delta A \end{aligned}$$

The new feature here is that to compute the slope $H'(t)$ you need to know both $H(t)$ AND $A(t)$. Similarly, to compute the slope $A'(t)$, you need to know both $A(t)$ and $H(t)$.

2. Apply Euler's Method to this model, using $\Delta t = 1/2$, to complete the first three rows of the following table:

t	$H(t)$	$A(t)$	Δt	$H'(t)$	$A'(t)$	ΔH	ΔA
0	84.98	3.5	0.5				
0.5			0.5				
1.0			0.5				
1.5			0.5				

3. According to your calculations, does the value of $A(t)$ initially increase or decrease?

4. By looking at the rate equations and by thinking about the experiment, what do you expect will happen to the *difference* between $H(t)$ and $A(t)$ as time passes?

Introducing True BASIC

You have learned enough Excel by this point that you could create a table in Excel to compute Euler's Method for this model. Rather than asking you to do so, however, this lab will introduce you to a program `COOLER2`, written in the programming language True BASIC, that already does this for you.

Go with your lab group to the computers and Click on the following icons: Mathematics → TrueBASIC Calculus → TBbronze→ Math110

Open COOLER2.TRU

Two windows will open up when you launch this program. One of them has the True BASIC program COOLER2. This is the window you will interact with to modify and run this program. As the course progresses, you will become somewhat more familiar with this programming language. **For now, just note where the rate equations and initial conditions are written in the program.**

This program uses Euler's method to calculate and plot piecewise linear approximations to $H(t)$ and $A(t)$. You can figure out what value of Δt is being used through the following calculation:

$$\frac{\texttt{t\_final} - \texttt{t\_initial}}{\texttt{number\_of\_steps}} =$$

5. Why can you find Δt this way? What value of Δt is being used?

With the mouse, select **Run** from the menu bar at the top of the window. When the program runs, it opens up a *graphics* window giving both graphs and a table of values. The program will pause part way through the calculations. Depress the space bar to have the program continue.

The graph of the Euler's Method approximation to the model solution for $H(t)$ appears in red, the graph of the Euler's Method approximation to the model solution for $A(t)$ appears in blue and the plot of the values of $H(t)$ observed experimentally appear in green. You will see both a graph and a table of values.

6. Do these results support your prediction concerning the difference between $H(t)$ and $A(t)$?

7. In what ways is this approximation to the experimentally observed values of $H(t)$ superior to the approximation from the simple Newton's Law of Cooling? In what ways is the revised model still not adequate?

Sensitivity to Parameter Values

8. With the mouse, click on the box at the upper left corner of the screen to return to the program. As was mentioned before, the parameter values were found mostly by trial and error. Pick at least TWO of the parameters listed below, and report on the effect of decreasing AND increasing their magnitudes while keeping the other parameters at their original values.

Decrease $A(0)$ to 0:

Increase $A(0)$ to 7:

Decrease k_1 to 0.005:

Increase k_1 to 0.008:

Decrease k_2 to 0.0005:

Increase k_2 to 0.0010:

Decrease k_3 to 0.001:

Increase k_3 to 0.00016:

Decrease k_4 to 0.001:

Increase k_4 to 0.0010:

Preparing Your Lab Report

Your report should consist of a cover page with the title of the report and the names and signatures of your lab group members. **Also indicate your Lab Section and Lab Time,** e.g. Section 3: Thursday 10:00am. Each person (in a group of three) should complete a first draft of one of the three parts, and the group should meet to read and discuss these drafts before submitting the final report. The final report is DUE IN NEXT WEEK'S LAB. Grading will be Credit/No Credit with, however, a high standard for receiving credit. If you do not initially receive credit you will have one week to revise the report to correct any problems with it.

You will need to refer to the lab handout for this lab to answer these questions.

Part 1

Pick *one* of the the equations for H'_A, H'_B, A'_H or A'_B and explain why the equation may be reasonable. Then state the full initial value problem which is the revised cooling model. In what ways does it differ from the simple Newton's Law of Cooling Model? Explain how Euler's method can be used to find piecewise linear approximations to $H(t)$ and $A(t)$. Plot these approximations for the first few steps (as you calculated in the lab).

Part 2

What, physically, do each of the parameters in this model represent? Report on your study of the sensitivity of solutions to changes in some of the paramters in this model. Interpret these changes in terms of the physical meaning of the parameters.

Part 3

This question may seem easier than the rest, but is deeper in some ways. Just because this model does a pretty good job of fitting the data, does this mean that it is nearly correct in accounting for the physical processes producing the cooling? Can you think of ways (perhaps looking at the relative magnitudes of some of the parameters) in which the model may be *incorrect*?