Objectives:

1. To introduce/review graphing procedures of the TI-83
2. To make connections between the symbolic derivations of extrema and the graphical presentations of extrema.
3. To review vocabulary of extrema.

Introduction

Optimization is the process of finding the extreme (i.e., maximum and minimum) values of a function. The derivative is an useful optimization tool: anywhere the derivative is zero is a natural candidate for a max or min. Some subtleties, however, do arise. A root of the derivative, for instance, might correspond to a local maximum, a local minimum, or neither. And a maximum or minimum, in turn, may be global or merely local.

To help sort things out, we'll work through several examples to review the language and the techniques of optimization.

§1. Vocabulary

We look for extrema of a function f at points where the derivative f' changes sign. Why? What is happening graphically when f' changes sign?

- If the derivative f' changes sign at $x = x_0$, then

  $$\text{either } f'(x_0) = 0 \text{ or } f'(x_0) \text{ does not exist}$$

  The value x_0 is a *critical point* of the function f.

  The <u>First Derivative Test</u> for finding Local Extrema

  If $f^{'}(x)$ is negative to the left of x_0 and positive to the right of x_0, then $f(x)$ has a local minimum at x_0.
  If $f^{'}(x)$ is positive to the left of c and negative to the right of x_0, then $f(x)$ has a local maximum at x_0.

  <u>Second Derivative Test for Local Extrema</u>

  If $f^{''}(x_0) > 0$ then $(x_0, f(x_0))$ is a local minimum.
  If $f^{''}(x_0) < 0$ then $(x_0, f(x_0))$ is a local maximum.
  If $f^{''}(x_0) = 0$ then $(x_0, f(x_0))$ is a possible inflection point.

- x_0 is a *local maximum* of f if $f(x_0) \geq f(x)$ for all x in some interval containing x_0.

Complete the following definitions (in your own words):

- x_0 is a *global maximum* of f if ...
- x_0 is a *local minimum* of f if ...
- x_0 is a *global minimum* of f if ...

§2. A First Example

Let $$f(x) = |x| \quad \text{where} \quad -\infty < x < \infty.$$

Sketch the graph of f below and complete the sentences that follow.

1. Does f have a local minimum? If so, where? Use your definition to defend your answer.

2. Does f have a global minimum? If so, where? Use your definition to defend your answer.

3. Does f have a local maximum? If so, where? Use your definition to defend your answer.

4. Does f have a global maximum? If so, where? Use your definition to defend your answer.

Now let $$g(x) = |x| \quad \text{where} \quad -3 < x \leq 2.$$

1. g *does/does not* (circle one) have a local maximum at $x = 2$ because ...

2. g *does/does not* (circle one) have a global maximum at $x = -3$ because ...

§3 Primer on Graphing with the TI-83

Introduction

If you already know how to graph using your calculator you can skip this section and go directly to the next Section.

In addition to basic scientific or statistical calculations, many calculators these days have great graphing capabilities. One of the reasons the mathematics department has chosen the TI-83 as the standard calculator for its calculus and statistics courses is that it contains powerful graphing capabilities which are fairly easy to access through basic menus on the calculator. This sheet serves as an introduction (or a review) of the basics of graphing functions on the TI-83. Further details on graphing can be found in Chapter 3 of the calculator Guidebook.

§3.1 Function and Graphing Modes

1. To work with and graph functions of the sort we will be discussing here, you need to make sure the calculator is in the right mode. When you press the `MODE` key, you will see a list of mode options. Basically everything on the left should be highlighted. In particular, we will look at three of the settings in detail.

   (a) On the fourth row, make sure `Func` is highlighted. This allows us to work with functions of the form `Y = f(X)`. Other function modes are parametric functions, functions given in polar coordinates, and sequences.

   (b) On the fifth row, choose `Connected` so that your final graphs will appear more like connected curves rather than dots along the curve.

   (c) And on the sixth row, choose `Sequential`. This will plot multiple functions one at a time whereas `Simul` will plot all functions at the same time as the calculator works through the given domain.

   You can exit out of the modes by pressing the `CLEAR` key or pressing the `2nd` and then `Quit` keys.

§3.2 Defining Your Function

1. We define the functions we wish to plot by first pressing the `Y=` key. If you are in the right function mode, you should see a line of `Plot1` through `Plot3` followed by places to define various `Y` functions. The `∧` and `∨` keys will allow you to move from one function to the other.

   (a) Try putting in a function, for example with the cursor blinking next to `Y1`, input $\sin(x)$ using the keys `SIN` `X,T,Θ,n` `)`. Note that when you press `ENTER`, the cursor moves to the next function, *and* as soon as you started typing, the equal sign for `Y1` becomes highlighted! This means that if you get around to graphing, this function will be graphed.

   (b) Let's put in another function for `Y2`. For example, `X,T,Θ,n` x^2 `−` `1`. (Note: Pressing x^2 alone will not place the variable x into the function expression.) Again, this function is automatically highlighted. To de-select the function (say you only want to graph `Y1` and not `Y2`), you would simply move the cursor to the left using the `<` key and hit enter when it is on top of the equal sign. You do the same to re-select the function. Make sure both are selected for the purposes of this exercise.

(c) If you ever want to delete a function that is already there, just make sure the cursor is on that line and press `CLEAR`.

§3.3 Setting the Window for your Graph

1. Before we actually graph the function, let's set up an appropriate window for our graph. To do this, press the `WINDOW` key. Note that there will already be values defined for each variable.

   (a) **Domain**. The first three values define what part of the domain of the function you will see when graphing. `Xmin` will define the left end of your graph, `Xmax` the right end, and `Xscl` how often tic marks will appear on your x-axis. For our examples above, let's define the domain to be $[-5, 5]$ with tic marks every one unit.

   (b) **Range**. The following three values define what part of the range of the function you will see when graphing. What are reasonable values to choose for our examples above?

§3.4 Graphing the Function(s) and Tracing

1. Now press `GRAPH` and you should see both functions graphed on a set of axes. One of the nice things you can do with the graphs is trace the functions, allowing one to find particular values of the function.

   (a) Press the `TRACE` key and you will see a flashing x-shaped cursor somewhere on the graph of one of the functions. In the upper left corner of the screen, you will see which function you are tracing; at the bottom of the screen, you will see the x and y coordinates of the point the cursor is on.

   (b) To move along the same function curve, use the `<` and `>` keys. To go from one function to another, use the `∧` or `∨` keys.

   (c) If you trace past the edge of the given domain, the calculator will shift the domain for you (but it won't do so for the range). You can see the change in the given domain if you go back to the `WINDOW` — the `Xmin` and `Xmax` have changed!

   Note that in the `GRAPH` window, there is also a free-moving cursor (rather than tracing the function, it moves anywhere on the window). You can move around the window when you have the functions graphed by simply hitting any of the four arrow keys.

§3.5 Changing the Viewing Window by Zooming

1. We can use the `ZOOM` key to redefine our viewing window in ten different ways. We will highlight a few of these. When you press this `ZOOM` key, you will see a menu of zooming options. You select one of these options by highlighting the number using your arrow keys `∧` or `∨` and then pressing `ENTER`, or by simply pressing the number key associated with the number option you wish to choose.

   (a) **ZBox**. This option allows you to redefine the window by drawing a box around the portion of the graph you want to expand. After selecting `1:ZBox`, the graph window reappears and the cursor blinks. Move the cursor to one corner of the rectangular region you wish to expand. Press `ENTER`. The cursor becomes a blinking box. Move the cursor to the opposite corner of your rectangular region. You will see the rectangle outlined with two small boxes on the two corners. Press `ENTER` again and the graph will be redrawn with the chosen rectangle as the window. The cursor is still blinking, thus you can choose another rectangle within this new window.

(b) **Zoom In**. This option will allow you to zoom in on the graph drawn. After selecting `2:Zoom In`, you will return to the graph window and see the cursor blinking. Move this to the point that you want to be the **center** of the zoomed in graph. Once you press `ENTER`, the graph will be redrawn. Since the cursor is still blinking, you can continue zooming in within this new window.

(c) **Zoom Out**. This option works the same way as `2:Zoom In`, but it will allow you to zoom out beyond the region drawn.

(d) **ZStandard**. This option automatically sets the window with domain $[-10, 10]$ and range $[-10, 10]$.

(e) **ZoomFit**. This option will redraw the graph so that the range fits exactly the range of the function over the domain already given. As soon as you select this option, the calculator take a few moments to calculate this range before redrawing the graph.

Note that everytime you change the window using the zoom options, the `Xmin`, `Xmax`, `Ymin`, and `Ymax` values in the `WINDOW` menu change accordingly.

§3.6 Pausing or Stopping a Graph

1. While plotting a graph, you can press `ENTER` to pause the drawing of the graph and then press `ENTER` to resume again.

2. While plotting a graph, you can press `ON` to stop the drawing of the graph and then press `GRAPH` again to redraw the whole thing.

§3.7 Other Graphing Options

1. There are many other graphing options on the TI-83 which you can explore on your own using the TI-83 Guidebook (beginning page references given in parenthases). These include:

   (a) setting various Graph Styles (3-9) for each function to be plotted (including shading above and below the graph of functions),

   (b) setting various Graph Formats (3-13),

   (c) storing Pictures (Pic) (3-3 & 8-17),

   (d) graphing a Family (or List) of Functions (3-16 & Ch. 11).

2. In addition, there are other types of functions you can define and graph. These include Parametric Graphs (Chapter 4), Polar Graphs (Chapter 5), and Sequences (Chapter 6).

§4. A Second Example with Graphs

Let

$$h(x) = \frac{x}{2} + \sin x \quad \text{where} \quad -\infty < x < \infty.$$

Graph this function on your TI-83 (use `ZOOM` `6:ZStandard` to set the plot window), and sketch it below.

1. How many local minima does h have?

2. Does h have a global minimum? Does h have a global maximum? Why or why not?

3. If the domain of h is restricted to $0 \leq x \leq 10$, will your answers to the previous two questions change? How?

4. If the domain of h is restricted to $0 < x < 10$, does h have a global minimum? Does h have a global maximum? Why?

We are demonstrating a nice property of continuous functions here.

> If a function f is continuous on a *closed* interval $[a, b]$, then f must have a global maximum and a global minimum on $[a, b]$.
>
> Furthermore, the global extremum occurs either at a critical point of f or at the endpoints of the interval $x = a$ or $x = b$.

Note that if a function is continuous on an *open* interval, it may or may not have a global max or min.

§5. Same Example with Symbols

Let

$$h(x) = \frac{x}{2} + \sin x \quad \text{where } 0 \leq x \leq 10.$$

1. Find all the critical points of h *exactly* (no decimal approximations allowed).

2. For each critical point, determine whether it is a local maximum or local minimum or neither. Defend your answer *without* the use of the graph.

3. What is the global minimum of h? What is the global maximum of h? Again, defend your answer *without* the use of the graph.

§6. Evaluating Critical Points

1. Let

$$k(x) = \pi + 2[\cos(x + \pi/3)]^3 \quad \text{where} \quad -\infty < x < \infty.$$

It can be shown that $k'(31\pi/6) = 0$, so k has a critical point at $x = 31\pi/6$. *Without* graphing k or evaluating k at any x-values or taking the derivative of k, can you conclude that k has at least a local max or min at $x = 31\pi/6$? Explain.

In sentences, write out a careful mathematical plan of attack to determine whether or not k has a local max or min at $x = 31\pi/6$, and then carry out the plan to reach a conclusion.

2. Let

$$j(x) = 2(x - 1)^{1/3} \quad \text{where} -\infty < x < \infty.$$

The function j has one critical point. Find it, and then determine whether the critical point is a local maximum, a local minimum or neither. Be sure to defend your answer.

Write-Up

The write-up for this lab involves you handing in a NEAT, LEGIBLY-WRITTEN LAB SHEET with the blank spaces filled in with the answers to the questions. You only need to hand in one solution sheet for each lab group. This lab sheet is due in lab next week: **Wednesday November 8/Thursday November 9**.