This lab will look at some ways that you can create new functions out of old ones. In the process, we will obtain some results that will be useful to us as we work with derivatives.

The Natural Logarithm Function

Consider the function

$$F(a,x) = \frac{a^x - 1}{x}, \qquad a > 0, \quad -\infty < x < +\infty.$$

This function actually has TWO input arguments. a and x. By taking the limit of this function as $x \to 0$, we will obtain a new function which is a function of a only. This new function will be the natural logarithm function. In fact, we will take this as the DEFINITION of the natural logarithm function!

Definition: The natural logarithm function, $\ln(a)$ is DEFINED as

$$\ln(a) := \lim_{x\to 0} \frac{a^x - 1}{x}, \qquad a > 0.$$

1. This definition depends on the fact that given a value of $a > 0$, the function whose formula is
$$y = \frac{a^x - 1}{x}$$
has a limit as $x \to 0$. To obtain evidence for this, modify the first `Wk08F00Lab` worksheet in *Excel* to estimate these limits for the values of a listed below. (The worksheet is already set up for $a = 0.5$.)

a:	0.5	2	3
$\lim_{x\to 0} F(a,x)$:			

2. You will notice that some of these limits are less than 1 and others are greater than 1. By trial and error with *Excel*, try to find a value of a such that $\lim_{x\to 0} F(a,x) = 1$. Record your best result below. Do you believe that such a value exists?

Definition: The constant e is defined to be that value of the base a for which $\ln(a) = 1$. In other words, e is defined *implicitly* by the property

$$\ln(e) = \lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

For values of x close to 0,

$$\frac{e^x - 1}{x} \approx 1 \quad \text{so} \quad e^x - 1 \approx x \quad \text{and} \quad e \approx (1+x)^{1/x}.$$

This approximation "becomes exact" in the limit as $x \to 0$, giving us an *explicit* formula for e:

$$e = \lim_{x \to 0} (1+x)^{1/x}.$$

3. Use the second `Wk08F00Lab` worksheet in *Excel* to estimate e using this limit. Record your best estimate below. Then take the estimate you found and use it for a in the first worksheet to confirm that $\ln(e) = 1$.

The Derivative of an Exponential Function

4. The results we have just obtained concerning the natural logarithm function are useful in determining a formula for the derivative of an exponential function. In the space to the right of the derivation below, give a reason for each step.

Let $f(x) = a^x$, where $a > 0$ is a constant base. Then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x \cdot (a^h - 1)}{h} \\ &= a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h} \\ &= a^x \cdot \ln(a). \end{aligned}$$

5. Use this result to determine the derivative of $g(x) = e^x$.

Composing a Function with a Linear Function

A common way to create a new function from old ones is *composition*. If we compose a function $f(y)$ with a linear function $y = g(x) = mx + b$, we get a new function

$$h(x) = f(g(x)) = f(mx + b).$$

The purpose of this part of the lab is to gain some insight into the way the graph of h is related to the graph of f.

6. Answer each of the following questions below. In each case, we recommend you experiment by taking $f(x) = \cos(x)$ and $f(x) = x^2$, taking $m = -1$, $1/2$ and 2, and taking $b = -2$ and 2. Use *Derive* or your graphing calculator. Support your answer with evidence from your experiments and with reasoning about how graphs and function composition work.

   How is the graph of $h(x) = f(mx)$ related to the graph of $f(x)$?

   How is the graph of $h(x) = f(x + b)$ related to the graph of $f(x)$?

   How is the graph of $h(x) = f(mx + b) = f(m \cdot (x + \frac{b}{m}))$ related to the graph of $f(x)$?

Further Notes

Preparing Your Lab Report

As before, your report should consist of a cover page with the title of the report, the names and signatures of your lab group members, and the DAY and TIME of your lab section. Each person (in a group of three) should draft one of the three parts.

Part 1

Confirm that

$$a^x b^x - 1 = (1 - a^x) \cdot (1 - b^x) + a^x - 1 + b^x - 1.$$

Then use this result, the definition of the natural logarithm function in this lab, and properties of limits to show that

$$\ln(a \cdot b) = \ln(a) + \ln(b).$$

How does this support the claim that the definition of $\ln(x)$ given in this lab is correct?

Part 2

State the best approximation to e that you found in this lab, and fully discuss the arguments and evidence you have that this approximation is a good one.

Part 3

How is the graph of $h(x) = f(mx + b) = f(m \cdot (x + \frac{b}{m}))$ related to the graph of $f(x)$? Give evidence and reasoning to support your answer, and be sure to consider all possible values for m and b.