Strengths and Weaknesses of Newton's Method

Faculty Information:

Worksheet Filename: `h:/mathwork/ti83exce/derivapp/newtmet/ti83/newtmet1.tex`

Objectives:

1. To review Newton's Method symbolically and graphically.
2. To use the TI-83 to perform the recursions of the method.
3. To explore some of the weaknesses of Newton's Method.
4. To observe the efficiency and robustness of Newton's Method.

File Dependencies:

1. `h:/mathwork/ti83exce/ti83xcel.sty`

Comments:

1. The first section gives instructions or Newton's Method on the TI-83.
2. The second and third sections depend on knowledge of the TI-83 commands.
3. Section 2 could be a used as a homework exploration to follow up on a presentation of Newton's Method in class.

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Introduction

Recall that Newton's Method is a recursive process constructed to find the roots or zeros of a function. We begin with an initial estimate of a root and then the method produces successive approximations of a root. In this worksheet, we will see how the TI-83 can do the recursive steps for us quickly. We will explore the ways in which the method can fail to find a root, and also observe how quickly the method can succeed at finding a root.

§1. A First Example with the TI-83

Let

$$f(x) = 3x^4 - 16x^3 + 6x^2 + 24x + 1.$$

How many roots could this function have? How do you know? On your TI-83, define the function as `Y1` in the `Y=` window. Then, in `WINDOW`, set the plot range for $-2 \leq x \leq 5$ and $-75 \leq y \leq 75$. Graph the function and record a rough estimate of the value of the roots of f.

From the graph, $x = 0$ looks like it is fairly close to a root of f. (How do you *know* it is not a root?) We'll use $x = 0$ as our initial estimate of the root, and get the TI-83 to improve the accuracy.

1. Find $f'(x)$.

2. Define `Y2` as this derivative function in the `Y=` window.

3. Recall that Newton's Method uses the recursive equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

so in the `Y=` window, we will define

`Y3 = X - Y1/Y2.`

How does this fit with the recursive equation above?

Note: To obtain the `Y1`, `Y2`, etc. symbols on the TI-83, press `VARS` and then press `>` to highlight `Y-Vars`. Choose option `1.Function...` by pressing `ENTER` and then select the `Y`-variable you need.

4. Using $x_0 = 0$ as our initial estimate of the root, find the value of x_1 by hand.

5. The TI-83 defines $0 - f(0)/f'(0)$ as $Y3(0)$. (Why?) So let's check your answer. On the home screen, press `0` `ENTER`. This temporarily stores the value 0 as `ANS` (see yellow command above `(−)`).

6. Now enter `Y3(ANS)`, and you should get $-.0416666667$. This is the first approximation that Newton's Method produces: $x_1 = -.0416666667$.

7. Plugging this x-value back into f and f' is not a joy by hand, but we are set up to do this quickly with the TI-83. The value $-.0416666667$ is now temporarily stored as `ANS`, so we merely want to find `Y3(ANS)` again. Since this is just a repetition of the previous command, all we have to do is press `ENTER`. Try it. You should see the value of x_2 on the screen.

8. Continue the process by pressing `ENTER` until you have feel that you have ten digits of accuracy. (How do you know?) Record the root you have found below.

9. Use the same process to find all the other roots of f, starting from your initial estimates obtained from the graph. Also note the number of iterations needed to obtain the ten digits of accuracy. Record your results in the following table.

Initial Estimate	Root	Number of Iterations

§2. Stumping Newton's Method

Let

$$g(x) = x^3 - 10x^2 + 22x + 6.$$

How many roots could this function have?

On your TI-83, graph the function g, adjusting the window until you are sure that you can see all the roots of g. Make a careful sketch of the graph below.

List rough estimates of the roots of g:

As before we'll use the TI-83 to find these roots with Newton's Method. In the $\boxed{\texttt{Y=}}$ window, define `Y1` to be g and `Y2` to be g'. Does `Y3` need to be adjusted? Why?

1. Suppose the initial estimate of a root of g is $x_0 = 2$. Which root of g would Newton's Method produce from this initial estimate?

   Draw on your sketch of the graph how Newton's Method would produce the approximation x_1.

   Use the TI-83 to find the next several successive approximations of the root produced by Newton's Method, and record these approximations on the next page:

   $x_0 = 2$

   $x_1 =$

   $x_2 =$

   $x_3 =$

   $x_4 =$

   $x_5 =$

   What do you notice? What is happening *graphically*? Will Newton's Method produce the root eventually?

2. The initial estimate $x_0 = 2$ did not work very well. Try again with $x_0 = 2.01$. Does Newton's Method work in this case? What root does the method find?

3. Suppose in trying to find a root of g, we use $x_0 = 5$. Set your TI-83 aside for a minute and answer the following: What root would Newton's Method be likely to produce? Will it in fact reach this root? How do you know?

4. Suppose this time $x_0 = 5.1$. Predict which root you think Newton's Method will produce. You may want to refer to your graph again.

   Now carry out the recursive calculations on your TI-83 to see what happens. Did you get the root you expected? Explain what is happening *graphically* starting from this value of x_0.

5. Suppose $x_0 = (10 + \sqrt{34})/3$. What happens when the TI-83 tries to do the recursive calculations? Why does this happen? Explain this both by referring to the recursive equation and to the graph of g.

6. From your explorations in this section, summarize the problems that Newton's Method can run into when you are trying to find a specific root of a function.

§3. Efficiency of Newton's Method

In the last section, we explored problems that the recursive method can have. Do not be misled to think that Newton's Method is a fragile or temperamental process. These problems result from particularly bad choices made for the initial estimate of a root. In this section, we will observe how powerful Newton's Method can be by seeing how quickly it can get an estimate of a root accurate to nine decimal places, even from rather poor initial estimates of the root.

Recall the function

$$g(x) = x^3 - 10x^2 + 22x + 6.$$

The function g has a root near $x = 6.5$. The exact value of this root is

$$x = \frac{10}{3} + \frac{2\sqrt{34}}{3} \cos\left[\frac{\pi}{6} + \frac{1}{3}\tan^{-1}\left(\frac{71\sqrt{47}}{1269}\right)\right].$$

Use your TI-83 to get a decimal approximation of this root accurate to nine decimal places, and record it below.

In the following, you will be given various initial guesses for the root and will use your calculator to carry out Newton's Method, recording the successive approximations and the number of digits that are correct in each approximation. (You can stop the recursive process when you have nine decimal places of accuracy.)

1. $x_0 = 7$.

Approximations of the root	# of Correct Digits
$x_0 = 7$	0
$x_1 =$	

2. $x_0 = 10$.

Approximations of the root	# of Correct Digits
$x_0 = 10$	0
$x_1 =$	

3. $x_0 = 25$.

Approximations of the root	# of Correct Digits
$x_0 = 25$	0
$x_1 =$	

4. From your observations in this section, discuss the following issues:

   - how well Newton's method works, even with a poor initial estimate (i.e., the robustness of the method),
   - how quickly Newton's method improves the accuracy of the estimate of a root (i.e., the efficiency of the method).