Worksheet 27: Monday April 15
Tessellations: Tiling The Plane

Definition

A **tiling of the plane** or **tesselation** is a pattern that covers the plane with non-overlapping figures A **periodic** tiling is one in which there exists at least two translations in non-parallel directions in which the tiling is mapped onto itself. A tiling which is not periodic can be **nonperiodic** or **aperiodic**. A **monohedral** tiling is one in which the pattern is formed from a single identical shape that is repeated.

QUESTION: Which regular polygons can tile the plane?
ANSWER: the interior angles of the figure must evenly divide 360 degrees.

Monohedral Tilings
There are only three edge-to-edge regular monohedral tilings of the plane: (1) square; (2) triangle and (3) hexagon.

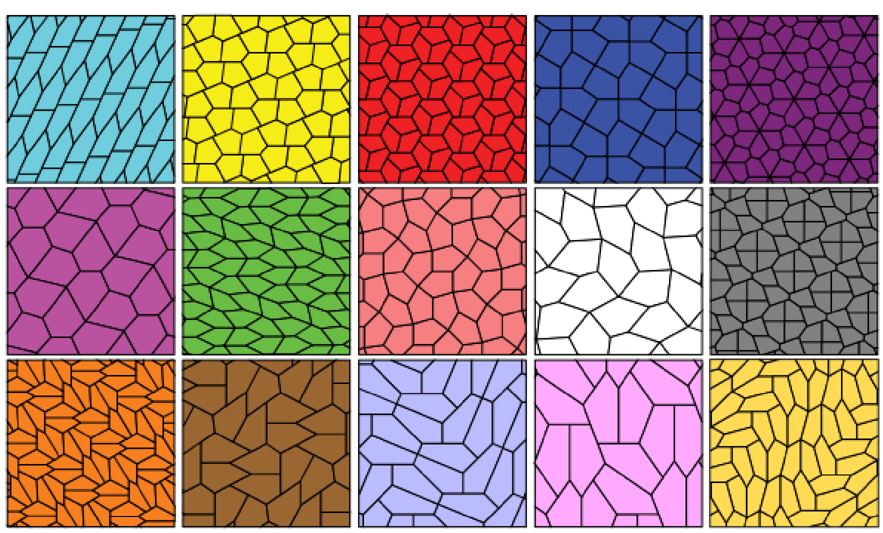
Any Parallelogram Can Tile The Plane!
Any parallelogram can tile the plane. You can lay the parallelograms side to side a form a strip. Then you lay the strips one on top of the other one to cover the plane.

Any Triangle Can Tile The Plane!
Any triangle (whether regular or not) can tile the plane. This is true because any triangle can be formed into a parallelogram by putting two copies of the triangle together. Note that there is a lack of reflection symmetry. We require **half-turns** (i.e. 180-degree rotations) of the figure to complete this tiling.

Any Quadrilateral Can Tile The Plane!
Any quadrilateral (not just squares and parallelograms) tiles the plane. This is true because you can create a hexagon with opposite sides parallel and congruent, and these can tile the plane. Again, we require **half-turns** of the figure (i.e. 180-degree rotations) to complete this tiling.

There Exist Irregular Polygons That can Tile The Plane!
German mathematician **Karl Reinhardt** (1895-1941) showed in his Ph.D. thesis that there are three types of irregular hexagons that can tile the plane.

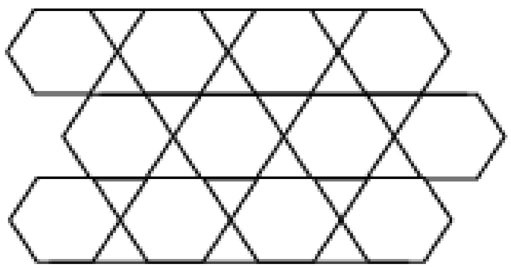
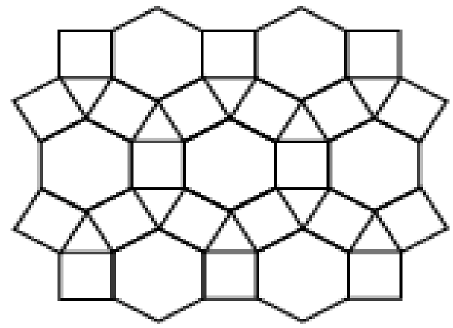
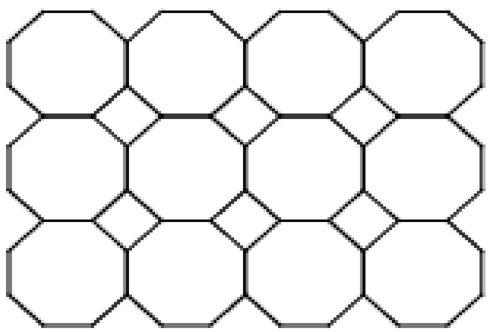
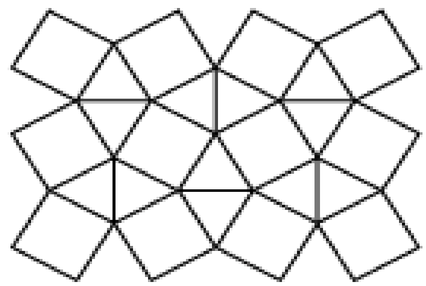
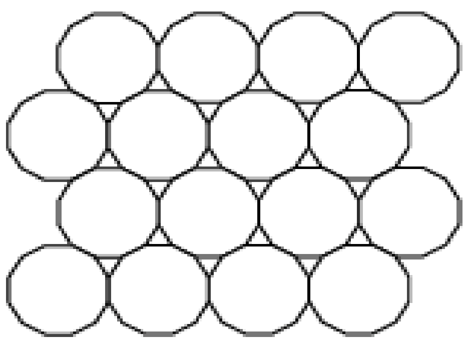
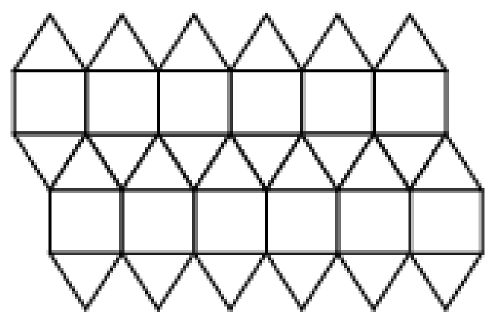
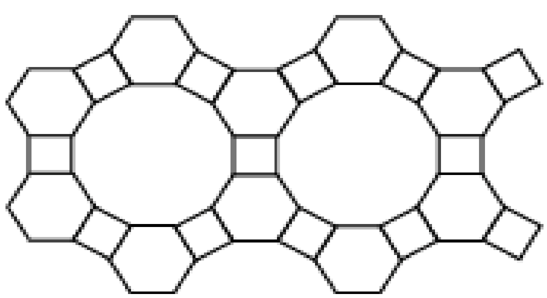
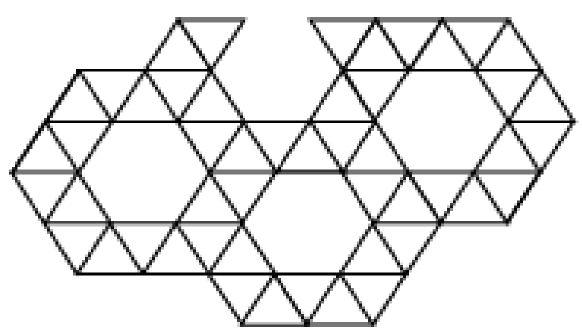
It used to be believed that there are 14 known types of irregular pentagons that can tile the plane. Four were found by Marjorie Rice, a San Diego housewife with no formal mathematical training beyond high school. But no one has proven whether this is all of them or not. Reinhardt found 5 (thought this was it, but couldn't prove it). ~~No new ones have been found since 1985.~~ In August 2015 a professor announced that she and her undergraduate students had discovered a 15th irregular pentagon tiling! (It's the one in the bottom right.)



No Convex Polygon With More Than Six Sides Can Tile The Plane
Reinhardt proved in 1927 that no convex polygon with more than six sides can tile the plane.

Semi-Regular Tilings
A semi-regular tiling is a tiling that uses two or more regular shapes and have the exact same behavior at each vertex. There are exactly **eight** “semi-regular” tilings.

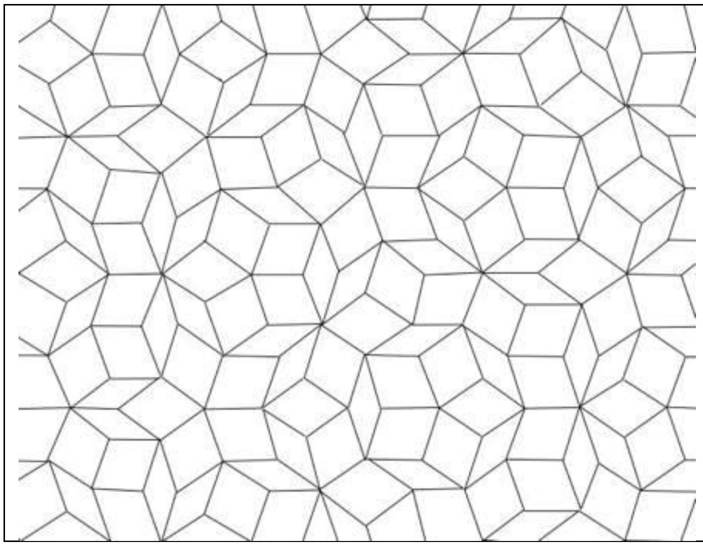
Can you identify which pairs of regular tilings are used in each case?

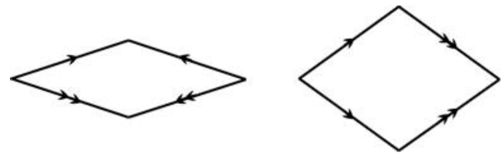
Note that all of the tilings of the plane we have looked at so far are periodic (i.e. they possess translational symmetries in two non-parallel directions).

Not All Tilings Are Periodic!

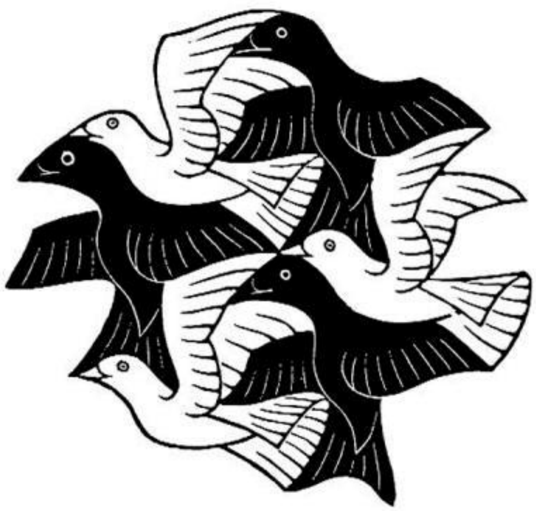
Roger Penrose (1931-) is a British mathematician who discovered a kind of aperiodic tiling (i.e. a tiling of the plane which only produces nonperiodic tiling consisting of only two “prototiles.” These kinds of tilings (seen below) are now known as Penrose tilings.



These are the “prototiles” of Penrose tilings known as the “thin rhomb” and “thick rhomb,” respectively. They have internal angles of 36 and 144 degrees for thin rhomb and 72 and 108 degrees for thick rhomb.



Replicating M.C. Escher



Now that we know about regular and irregular monohedral tilings, let’s see how we can make “art” out of them as is done by M.C. Escher and others as shown above.

Periodic Tiling Algorithm

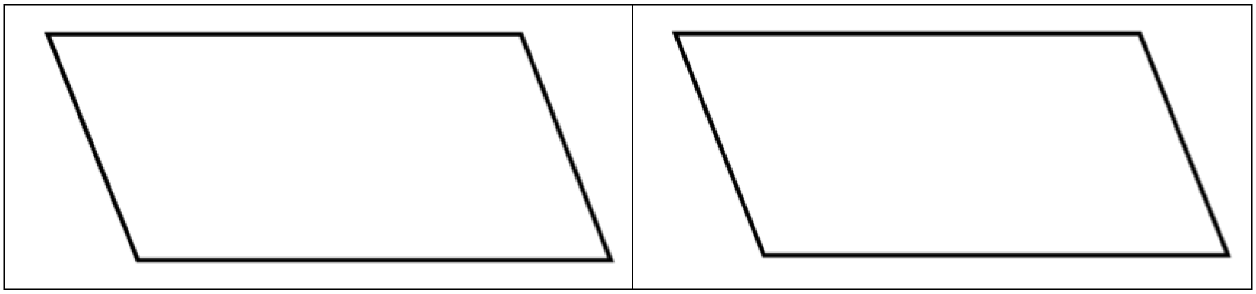
Begin with a tiling with a parallelogram.

Modify the parallelogram tiles but maintain it as a monohedral tiling by ensuring that translation of the tiles in both a “vertical” and a “horizontal” direction hold.

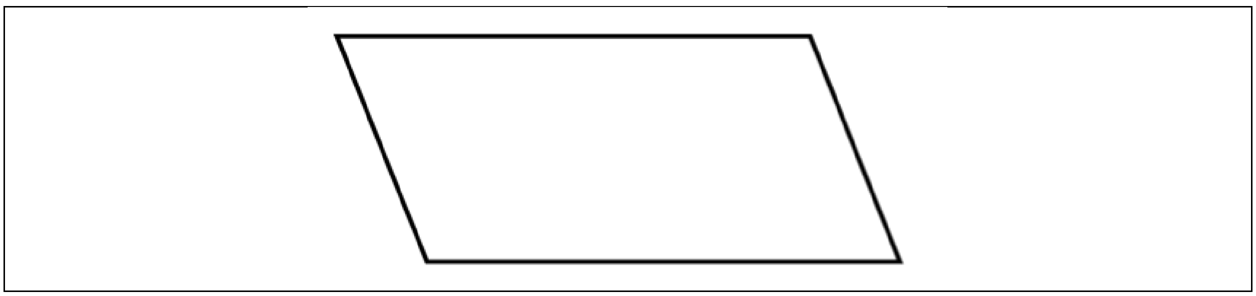
So we simply need to:

1. Modify a side and translate it to the opposite side.
2. Modify one of the other sides and translate it to the opposite side.

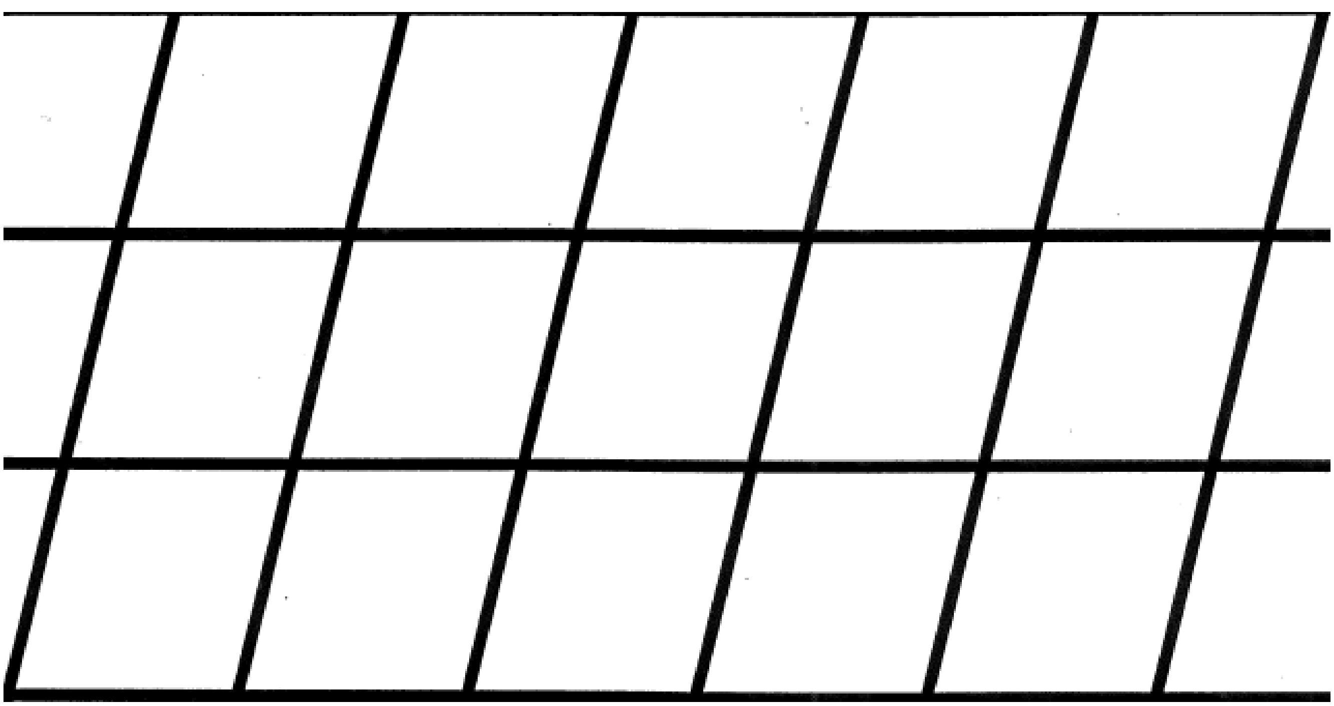
Make Your Own Period Tiling!
For example, we start with a parallelogram and then whatever change we make to one side we translate that change to the side paralle to it. And then we translate our new shape in the two directions that the parallelogram is symmetric in.



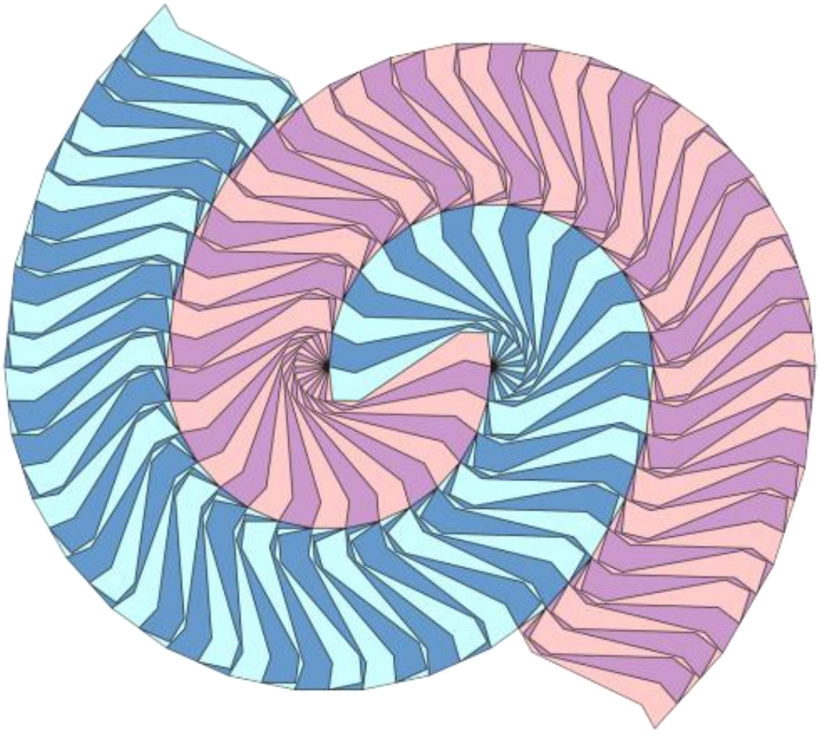
First Make Your Own Irregular Tile



Once you have made your desired alteration to the parallelogram, translate your new shape in the two directions that the parallelogram is symmetric an infinite amount of times to create a new periodic tiling!



The Voderberg Tiling: A monohedral non-periodic tiling of the plane by a 9-sided irregular polygon (nonagon).



The Einstein Tiling: A monohedral non-periodic tiling of the plane by a 13-sided irregular polygon (tridecagon). ("einstein" comes from German for "ein Stein" or one stone.)