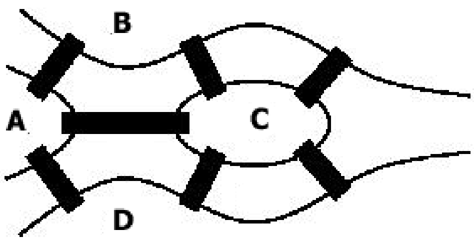
Class 26: Friday April 12
Euler and Topology

The Konigsberg Problem: The Foundation of Topology

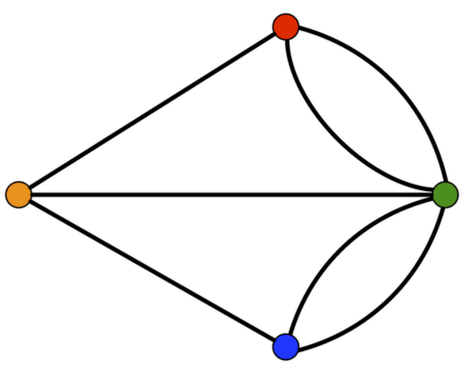
The Konigsberg Bridge Problem is a very famous problem solved by Euler in 1735. The process he used is considered to be the beginning of the mathematical subject of topology.
Below is a picture of the bridges connecting the land masses in Konigsberg (now Kaliningrad).



The bridges of Konigsberg

Here is the question: *Is there a route we can take that crosses each bridge exactly once*?

First, represent the land masses as vertices, and the bridges as edges so that we can reduce the problem to a network problem.



The question now becomes: *Is there a path that follows each edge only once?*

If we need to go in and out of each vertex (except for when we begin and end), can you deduce something about the number of edges that these vertices must have?

Have a look at your graph. Is it possible to travel through Konigsberg, crossing each bridge exactly once?

Definition **Eulerian Path:** A connected graph in which one can visit every edge exactly once is said to possess an **Eulerian path** or **Eulerian trail**.

Definition **Eulerian Circuit:** An **Eulerian circuit** is an Eulerian trail where one starts and ends at the same vertex.

Euler's Graph Theorems

Theorem 1: Euler circuits

If every vertex in a connected graph in the plane is of even degree (i.e. has an even number of edges coming out of it) **then** it must have an Eulerian circuit. **If** a graph has any vertices of odd degree **then** it can not have an Eulerian circuit.

Theorem 2: Euler paths

If a connected graph has more than 2 vertices of odd degree **then** it can not have an Eulerian path. **If** a connected graph has exactly 2 vertices of odd degree **then** it has at least one Eulerian path.

Theorem 3: Degrees of Graphs

The sum of the degrees of the vertices of a graph is an even number (twice the number of edges). The number of vertices of odd degree in a graph is always even.

Summarizing Euler's Graph Theorems

The number of vertices of odd degree determines what you can conclude

Number of Vertices With Odd Degree	**Implication from Euler's Theorems**
0	There is atleast one Eulerian circuit
2	There is at least one Euler path (and no Euler circuit)
$2k$ (where $k > 1$)	There are no Euler circuits or Euler paths.

EXAMPLE

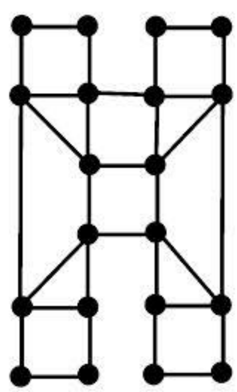
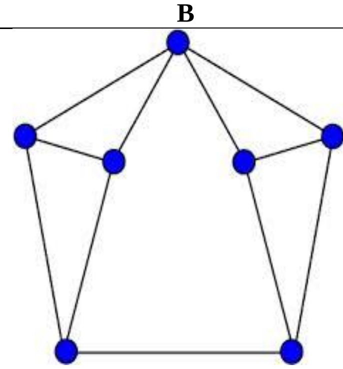
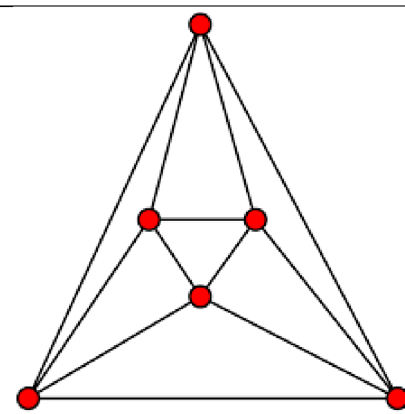
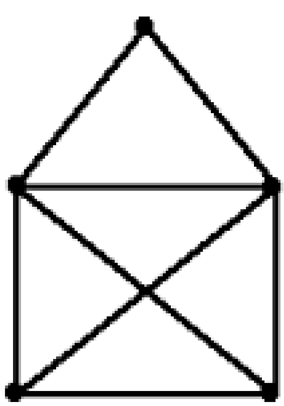
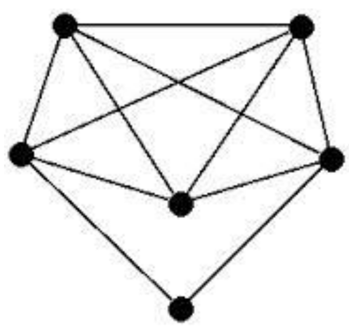
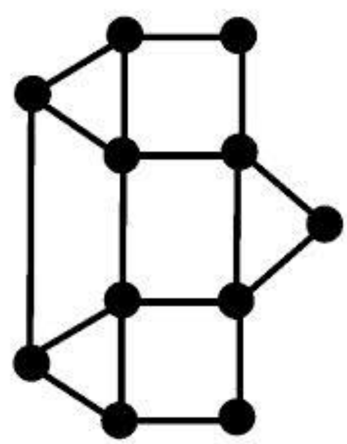
Let's demonstrate each of the three implications from the table with appropriate graphs.

GroupWork

Determine which (if any) of the following graphs must have atleast one Eulerian cycle.

Determine which (if any) of the following graphs must have atleast one Eulerian path.

Determine which (if any) of the following graphs must NOT have an Eulerian cyle or Eulerian path.

A	**B**	**C**
		
D	**E**	**F**
		

Exercise

For fun you can confirm the Eulerian characteristic $V\text{-}E\text{+}F\text{=}1$ for each of the given connected graphs in the plane.

For fun you can confirm Euler's Degree Theorem for each of the given connected graphs in the plane. (The sum of degrees of all the vertices in each graph equals twice the number of edges.)