Class 25: Wednesday April 9
Graphs and Euler

"Read Euler, read Euler, He is The master of us all."—Pierre-Simon Laplace (1749-1827)
Leonhard Euler (1707-1783) of Switzerland, in addition to being associated with the most famous equation in mathematics, $e^{i\pi} + 1 = 0$, was one of the most prolific contributors to mathematics of all time. He basically founded the mathematical subjects of graph theory (the study of graphs) and topology (the study of shapes). He is credited with writing more than 900(!) mathematical articles in his lifetime.

1. In the space below, place your pen on the sheet (mark it with a dot), close your eyes and draw a random curve, looping around and through 2 or three times, up and down, and mark where you end. Don't lift your pen and draw something with a little complexity, but not too much because we're going to do some counting. Because you have not lifted your pen, your drawing is a **connected graph**.

DEFINITION: connected graph

A connected graph is a graph where there is at least one edge between every pair of vertices.

2. Now accentuate each place where the curves cross each other in your drawing by placing a big dot at each crossing point. These are your **vertices**. Also, make sure that where you began and finished also have vertices. Your drawing should divide the paper into various regions (which we will call "**faces**"). The **degree** of a vertex is number of edges that go through it.

3. Except for your beginning and ending dots, how many curves come out of each of your other dots? Why do you think this is so?

4. Now count your vertices (dots) and number your edges (segments between vertices) and faces (regions completely surrounded by edges).

How many vertices? V =

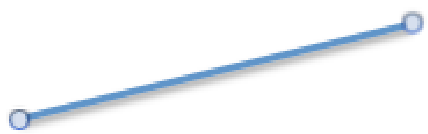
How many edges? E =

How many faces? F =

Let's record a variety of V, E, and F values from people in the class who have all drawn different curves. Let's see if we can find a relationship between vertices, edges and faces!

This relationship is a consequence of the *topology* of the paper (and the plane) so everyone's in the class will be the same!

5. Why is this so? Let's build up a picture. Here is the most simple of pictures with one edge.



2 Vertices, 1 edge, 0 faces
2 – 1 + 0 = 1

Now draw another line segment that starts at one vertex, goes in any direction, and ends in a new vertex. Count the vertices, edges, and regions. Does the relationship found in 4 still work?

Now join up the two end vertices to make a triangle. Count again. Does the relationship found in 4 still work?

On the previous page, continue adding segments that start at an existing vertex and either (1) go to an existing vertex or (2) just stick off someway and end with a new vertex. New edges may not cross or touch any point of the existing diagram except at an existing vertex. Each time, count again and see if the relationship found in 4 still works.

If you add an edge that connects two existing vertices, what increases and what stays the same?

If you add an edge that just sticks out somewhere ending in a new vertex, what increases and what stays the same?

So, whenever we add an edge (which subtracts one in the relationship found in 4), we also add either one more face or one more vertex (which adds one in the relationship found in 4). The objects you are drawing consist of edges connected together. They are connected graphs and we have basically shown the following:

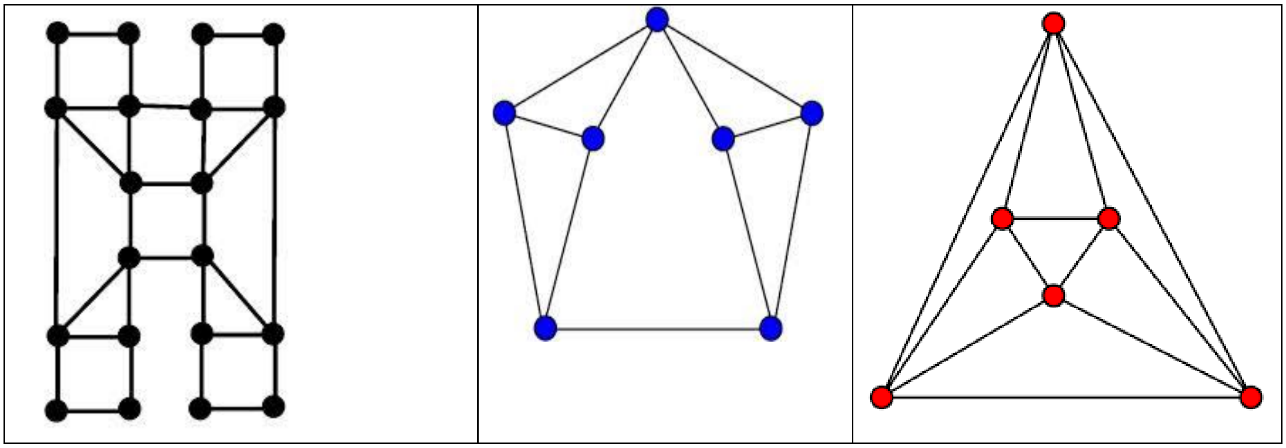
> **THE EULER CHARACTERISTIC**: For any connected graph in the plane,
> **V – E + F = 1**, where V is the number of vertices, E is the number of edges, and F is the number of faces.

Note that bending, stretching, twisting, shrinking the plane would not affect this result. Therefore this is a topological characteristic of the plane.

GroupWork

6. Verify that the Euler characteristic holds on the following graphs:
Calculate V, E and F in each case and calculate the Euler characteristic **V-E+F**.

a) b) c)



Exercise

Is it possible to draw a connected graph in the plane with 3 vertices, 5 edges, and 3 faces? If so, draw one below; if not, explain why not.

Is it possible to draw a connected graph on the plane with an even number of faces, an even number of vertices, and an even number of edges? If so, draw one below; if not, explain why not.