Class 23: Wednesday April 3

Euclid, Geometry and the Platonic Solids

Euclid's *Elements*

One of the most influential mathematical texts of all time is Euclid's *Elements*. *Elements* was published in 13(!) volumes (called books) by **Euclid of Alexandria** in approximately 300 BCE. It is a nearly comprehensive collection of Greek knowledge of geometry (which comes from the Greek word "earth measurement") and other mathematical topics at the time. It is thought to be only second to the Bible in the number of different editions that were (and are) circulated. Euclid's *Elements* is also famous for its approach to proof and the style in which statements of mathematical fact were made in the form of definitions, axioms (assumed to be true), propositions (proven via theorems and geometrical constructions) and proofs.

`DEFINITION`: **postulate**

A **postulate** or **axiom** is a mathematical statement which is so obviously true that it does not need to be proved and can be relied upon to be used to prove other statements.

The Five Axioms of Euclidean Geometry

Below are the five postulates (axioms) of Euclidean Geometry. We shall illustrate each with a sketch.

Postulate 1 It is possible to draw a straight line from any point to another.

Postulate 2 It is possible to produce a finite straight line continuously in a straight line (i.e., extend the line "infinitely" on both sides).

Postulate 3 It is possible to describe a circle with any center and radius.

Postulate 4 All right angles are equal to each other.

Postulate 5 Two non-parallel lines meet at a point.

NOTE The above statement of Postulate 5 is a bit different (and simpler) than the actual text of Euclid's :

> Parallel straight lines are straight lines, that being in the same plane and being produced [i.e. extended] indefinitely in both directions, do not meet one another in either direction.

All of these postulates may seem "obvious" to us. That is why they are accepted and why they form the underlying postulates of Euclidean Geometry.

The Controversial Fifth Postulate and Non-Euclidean Geometry

The first four postulates turn out to be necessary to produce a consistent geometry. However it turns out that the fifth one given is not necessary to produce a consistent geometry. There are other geometries that satisfy the first four postulates, but not the fifth, and are consistent geometries. There are three possible variations of this postulate that create consistent geometries. **Carl Friedrich Gauss** (1777-1855) and **Bernhard Riemann** (1826-1866) are the people most closely associated with what are called "non-Euclidean Geometries."

Illustrate each possibility with a sketch.

1. There is exactly one parallel.

This is what we are used to as the fifth postulate and what gives us Euclidean Geometry.

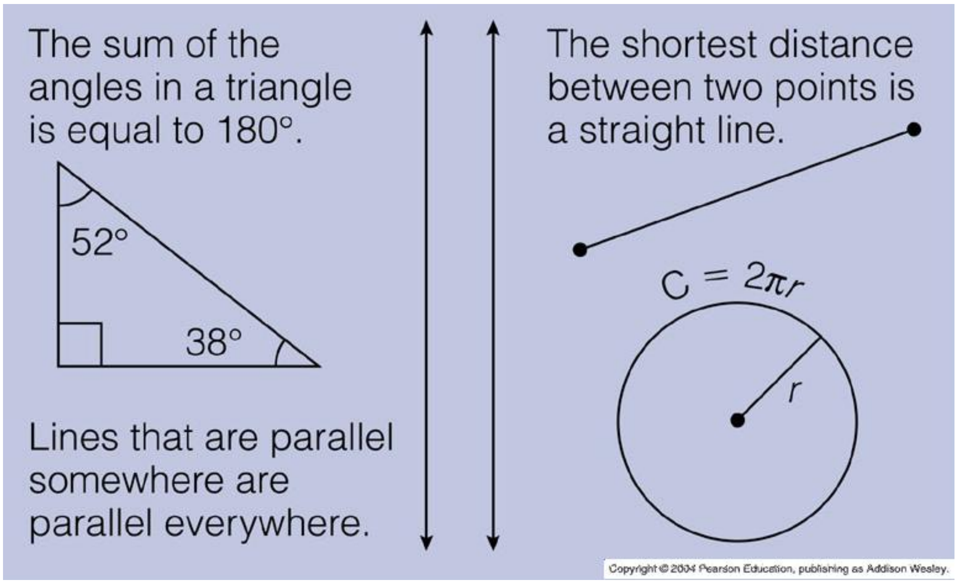
2. There are no parallels. [Stated another way: every two lines intersect.]

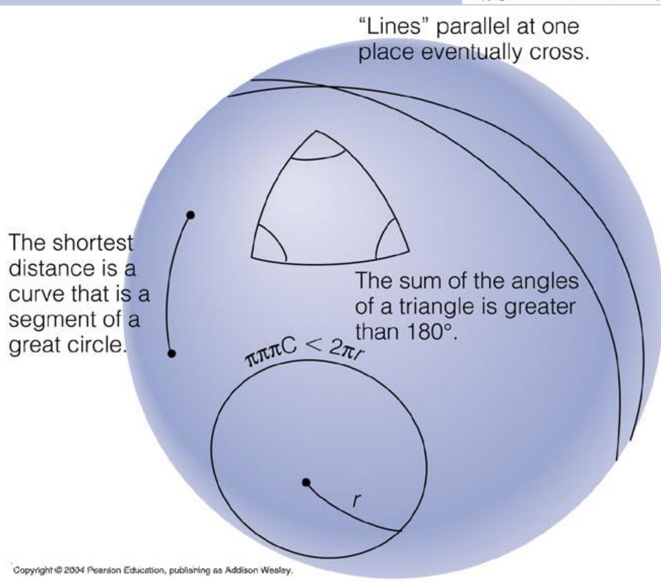
Using this statement as a fifth postulate leads to what we call Spherical Geometry.

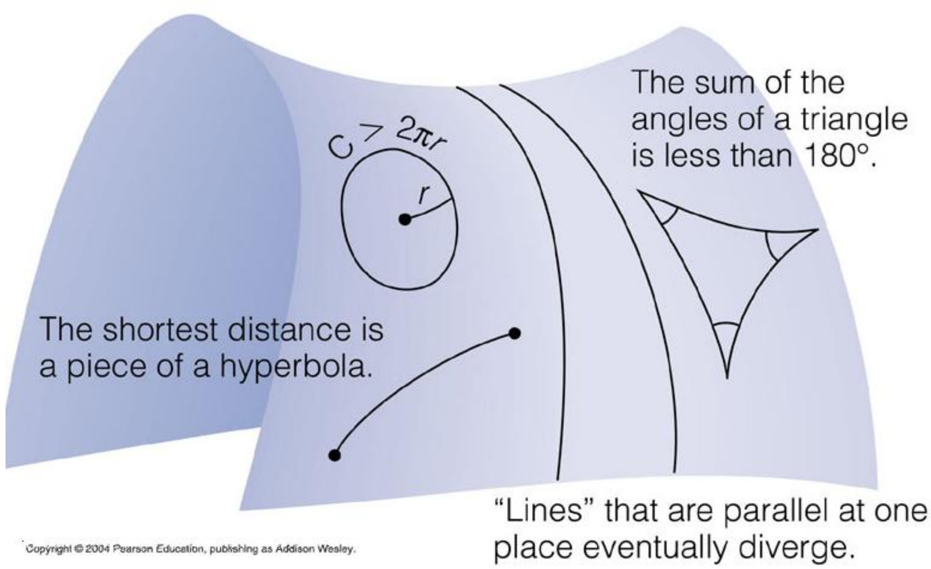
3. There is more than one parallel.

Using this as a fifth postulate leads to what we call Hyperbolic Geometry.

Non-Euclidean Geometries







The Platonic Solids

In the final of book of Euclid's *Elements* (Book XIII) he includes 18 propositions about regular solids. These solids are three-dimensional figures with planar faces where each face is a regular polygon and all faces are congruent, and all angles between pairs of adjacent faces are the same.

The Greeks had long known that there were only five such objects, and **Plato** (*circa* 428-348 BCE) discusses them in length in his work the *Timaeus*.
The Platonic Solids are:

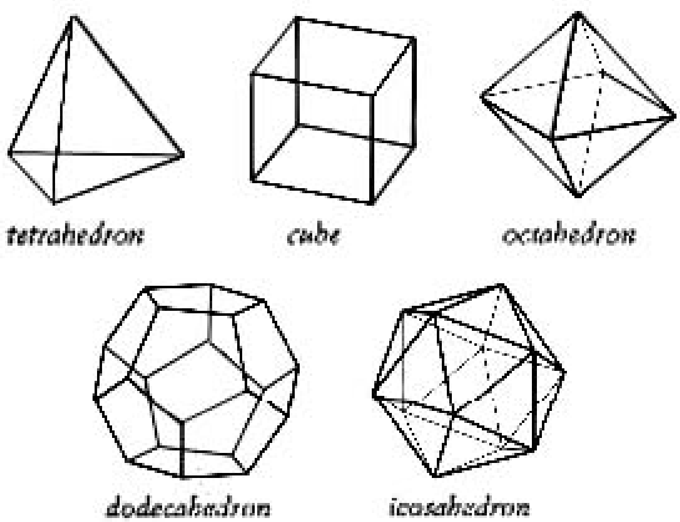
tetrahedron which has four faces, each one an equilateral triangle

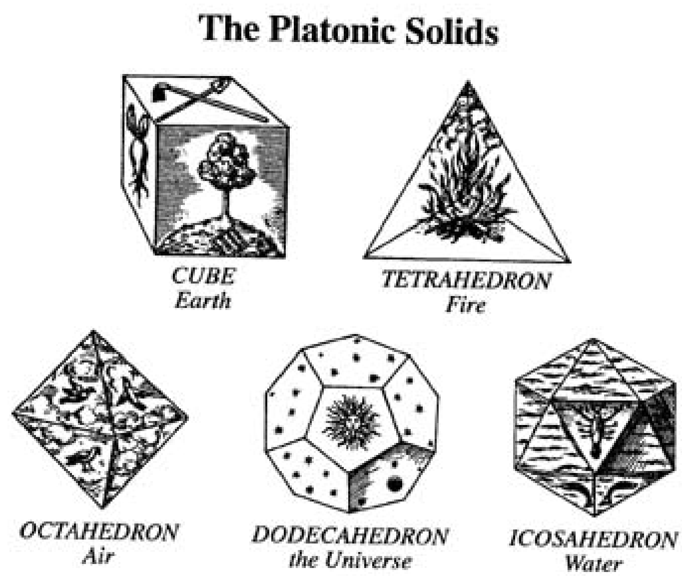
cube which has six faces, each one a square

octohedron which has eight faces, each one an equilateral triangle

dodecahedron which has twelve faces, each one a regular pentagon

icosahedron which has twenty faces, each one an equilateral triangle





Plato believed that the world must consist of elements made from these "perfect objects." The elements were Air, Fire, Water and Earth and each element had an associated Solid.

Analyzing The Five Platonic Solids

	Tetrahedron	**Cube**	**Octahedron**	**Dodecahedron**	**Icosahedron**
Faces are:					
# of Faces					

Proving Euclid's Last Proposition: There Are Only Five Platonic Solids

The very last proposition proved by Euclid in *Elements* (#865!) is that there are only five Platonic Solids. We will repeat this proof below.
What are the possible planar faces that a Platonic Solid can be made of? How do we prove there are no other possibilities? Let us consider the possibilities by first considering the regular polygons, since these shapes are combined to form the Platonic solids.

Exercise

How do we find the interior angles of regular polygons? What ARE the regular polygons? Draw the first several below and record their interior angles

In order to see which regular polygons can serve as a face of a Platonic Solid we need the following two conditions:

1. The sum of the angles at a vertex must be less than 360 degrees (Euclid's Prop. XI.21)

1. At least three faces meet at each vertex.

EXAMPLE

So how many of which regular polygons can make up the faces of a Platonic Solid?

Dual Platonic Solids

Fill out the table below and see if you can find any patterns or relationships between the numbers that describe the Platonic solids.

GroupWork

Platonic Solid	# of Faces	# of Vertices	# of Edges	# of Faces that meet at each Vertex	# of Sides of each Face
Tetrahedron					
Cube					
Octahedron					
Dodecahedron					
Icosahedron					

Which pairs of solids have the same number of edges? (These pairs are known as the Dual Platonic Solids.)

What's the relationship between the number of faces and the number of vertices for each of these pairs of solids?

What's the relationship between the number of faces that meet at each vertex and the number of sides of each face for these pairs of solids?

The Dual Platonic Solids are the ______________ and ______________ as well as the ______________ and ______________.

QUESTION **Which platonic solid is dual with itself?**