Class 22: Monday April 1

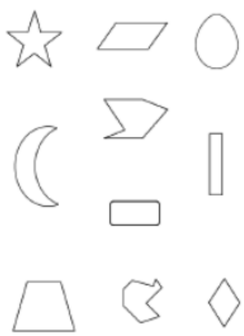
Introduction To Symmetry and Shapes

DEFINITION: **Symmetry**

An object is said to possess a **symmetry** (i.e., to be symmetric) if there exists a transformation that when applied to the object leaves it invariant (i.e., unchanged).

EXAMPLE

Which of the following shapes are symmetric?



Transformational Operations: Rotation, Reflection and Translation

Three common transformational operations are reflection, rotation, and translation.
To determine whether something has rotational symmetry, reflective symmetry or translational symmetry consider the following questions:

Reflection: When you flip a given object across some line (often called the *axis of reflection*), does it change?
If the answer to this question is **no**, the object is "symmetric under reflection" (possesses reflective symmetry).

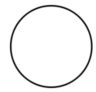
Rotation: When you rotate the given object over some angle about a point (often called the *center of rotation*), does it change?
If the answer to this question is **no**, the object is "symmetric under rotation" (possesses rotational symmetry).

Translation: Beginning with two copies of the same object that are superimposed (one on top of the other), can you glide or slide one of them (without rotating or reflecting) so that it lands in a new place where all the lines of the objects still match?
If the answer to this question is **no**, the object is "symmetric under translation" (possesses translational symmetry).

Other Transformations There are other transformations that exist that one can apply to objects. The most obvious is **Scaling** which stretches or shrinks the object, but by definition a scaling transformation can never be symmetric.

The **Identity** transformation basically takes the object and does nothing to it, so by definition the identity transformation is a symmetry.

Symmetries of the Circle
Consider the circle.



`QUESTION`: Is the circle symmetric under translation?

`QUESTION`: Is the circle symmetric under rotation around its center, say counterclockwise through 30, 60 or 90 degrees?

`QUESTION`: Is the circle symmetric under reflection about any diameter?

Compositions of Transformations
We can combine transformations. That is, we could apply a rotation and then a reflection, or apply one rotation and then another rotation, etc. This is called a composition of transformations. Suppose we have two transformations T_1 and T_2 that we apply to some object X to produce an image Y. If we apply transformation T_1 to X and then T_2 to that output we say that we have composed T_2 and T_1 together, i.e. $T_2 \circ T_1$ or $Y = T_2(T_1(X))$.
How do you "undo" a rotation?
How do you "undo" a reflection?
These transformations that undo a given transformation are called inverses of the given operation.

`DEFINITION`: **Inverse Transformation**

An **inverse transformation** S of a given transformation T is a transformation such that when S and T applied one after the other the result is the identity transformation $\mathcal{I}$. In other words,

$$S \circ T = T \circ S = \mathcal{I}$$

Sometimes S is denoted T^{-1} to say it is the "inverse of T."
The circle has rotational and reflective (or reflectional) symmetry. The set of these transformation form an object we call a group.

Groups
Group theory is the branch of mathematics devoted to the study of group. Names often associated with Group Theory is **Evariste Galois** (1811-1832), who coined the term in a paper published in 1829 (at age 18!). Other famous contributors to group theory are **Felix Klein** (1849-1925), **Arthur Cayley** (1821-1895), **Sorphus Lie** (1842-1899) and **Camille Jordan** (1838-1922).

`DEFINITION`: **Group**

A group G is an algebraic structure that combines a set of objects with an operation (often denoted $*$) that combines any two elements of the set to form a third element in the group. In particular we will look at **symmetry groups** which have the following properties

(1) The operation $*$ is associative (gives the same results regardless of the order of operation, i.e. $A * B = B * A$)

(2) There exists an identity element $\mathcal{E}$ in G such that an element $X * \mathcal{E} = \mathcal{E} * X = X$ for every element X in the group G

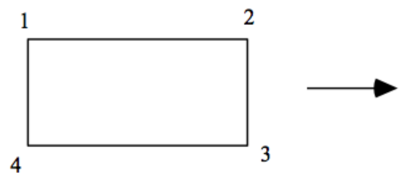
(3) For every X in the group G there exists an inverse element Y so that $X * Y = Y * X = \mathcal{E}$

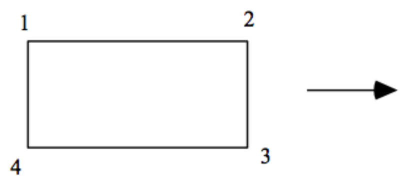
Symmetries of the Rectangle

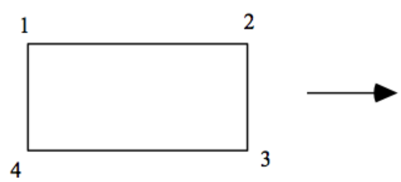
GroupWork

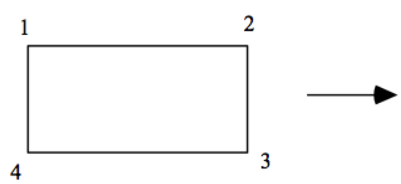
I have labeled the vertices of the rectangle in order to keep track of it. Think about where the vertices go when we either rotate or reflect. Draw the new rectangle next to it and where the vertices end up. Let's start with the easiest, which is the identity operation $\mathcal{I}$. How many others will there be? Which will be symmetries (i.e. symmetric transformations)?

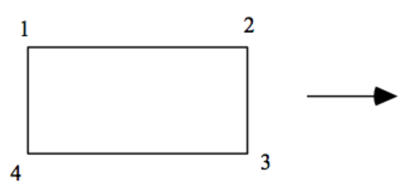
INPUT SHAPE OPERATION OUTPUT SHAPE SYMMETRY (YES/NO)

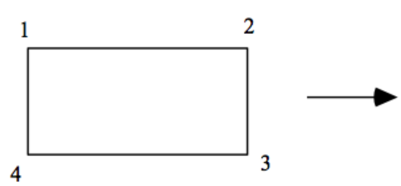












Any others that you can think of?

The Symmetry Group of The Rectangle

Let's label our symmetries for the rectangle we found on the previous page using the following identifiers. Does each of these symmetries have an inverse? What are they?
Write the inverse operation for each of the identified operations below. (Describe each in a short phrase.)
Identity $=\mathcal{I}$:

$\mathcal{I}^{-1} =$

Rotation clockwise by 180 degrees $= \mathcal{R}$:

$\mathcal{R}^{-1}=$

Reflection across the vertical $= \mathcal{V}$:

$\mathcal{V}^{-1}=$

Reflection across the horizontal $= \mathcal{H}$

$\mathcal{H}^{-1}=$

This is a VERY unusual situation! An operation whose inverse is itself is called **involutory**. It is very rare to have multiple involutory operations associated with a shape.

GroupWork

Let's do a reflection about the horizontal followed by a reflection around the vertical. Is this equivalent to any other transformation?
$\mathcal{V} \circ \mathcal{H} =$

Now do a reflection about the horizontal followed by a rotation of 180 degrees. Is this equivalent to any other transformation?
$\mathcal{R} \circ \mathcal{H} =$

What happens if we apply one of our transformations, say $\mathcal{R}$ twice in a row. Is this equivalent to any other transformation?
$\mathcal{R} \circ \mathcal{R} =$

We can record the results of all possible compositions in a "multiplication" table for our four operations $\mathcal{I}, \mathcal{V}, \mathcal{H}$ and $\mathcal{R}$ called a Cayley Table.

o	I	R	H	V
I				
R				
H				
V				

These four symmetries $(\mathcal{I}, \mathcal{R}, \mathcal{V}, \mathcal{H})$ form a group called **the symmetry group of the rectangle**. It is also called the Klein Group. (This is not a public relations firm!). Mathematically, this is a group with the set of transformations $(\mathcal{I}, \mathcal{R}, \mathcal{V}, \mathcal{H})$ as members of the set G and **composition** (i.e. do one operation then another) as the operation $*$.