Mathematics As A Liberal Art

Math 105 Spring 2024 Fowler 309 MWF 3:00pm- 3:55pm

http://sites.oxy.edu/ron/math/105/24/

Class 21: Friday March 29

Aleph One ($\aleph_1$) and All That

THEOREM Cantor's Power Set Theorem

For **any** set $\mathcal{S}$ (finite or infinite), the cardinality of the power set of $\mathcal{S}$, i.e. $P(\mathcal{S})$ is always strictly greater than the cardinality of $\mathcal{S}$.
Mathematically, this says that

$$|P(\mathcal{S})| > |\mathcal{S}|$$

Introducing $\aleph_1$ (Aleph One)

We have shown that the cardinality of the real numbers is greater than the the cardinality of the natural numbers. This result was proved in an 1891 paper by Cantor using a **diagonalization argument** in his proof. We know that the cardinality of the natural numbers is $\aleph_0$. We call the cardinality of the real numbers $\mathbf{c}$.

Now, if we apply the formula for the cardinality of the power set to the set of natural numbers,

$$|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0} = \aleph_1$$

The cardinality of the power set of the natural numbers is called $\aleph_1$. In theory, we could repeat this process and obtain $\aleph_2 = 2^{\aleph_1}$. These numbers would all be larger than $\aleph_0$ (i.e. the infinity that represent the cardinality of the set of natural numbers). They are called **transfinite numbers**.

QUESTION: What are the relative sizes of $\aleph_0$, $\mathbf{c}$ and $\aleph_1$?

`THEOREM` The Continuum Hypothesis, i.e. $\aleph_1 = \mathbf{c}$

The continuum hypothesis is that the cardinality of the continuum (i.e. the set of points in a finite line segment) which is sometimes denoted **c** is exactly equal to $\aleph_1$. This idea is somewhat controversial in some mathematical circles. There are some that believe that (the continuum hypothesis is FALSE) and there are many transfinite numbers between $\aleph_1$ and **c** and there are some that believe (the continuum hypothesis is TRUE) so these two numbers are the same.

However all mathematicians believe there are an infinite number of infinities. All we have to do is think about the expression $2^{\aleph_1}$, which is presumably the cardinality of the power set of the power set of the natural numbers, or $|\mathcal{P}(\mathcal{P}(\mathbb{N}))|$ and then continue this process over and over again to convince yourself there are more transfinite numbers.

Kurt Gödel (1906-1978) proved (in what is called **Godel's Incompleteness Theorems** that if the continuum hypothesis is true then it is consistent with all the other postulates that make up set theory and the rest of mathematics. It has also been shown that the rest of mathematics is consistent if the continuum hypothesis is false. In other words, whether or not the continuum hypothesis is true or false has no impact on the rest of mathematics as most people know it or understand it!

Regardless, even though not all mathematicians are convinced that the Continuum Hypothesis is true, after Cantor all mathematicians are convinced that there are definitely multiple infinities.
The mathematical statement of the continuum hypothesis is

$$2^{\aleph_0} = \mathbf{c}$$

QUESTION: Does it surprise you to discover that there are statements (like the continuum hypothesis) which can not be determined to be either true or false?

The Liar's Paradox

How about this statement which is known as the Liar's Paradox: **This statement is false.**