Class 20: Wednesday March 27

Infinite Sets, Power Sets, Bijections and Cardinality

DEFINITION: One-To-One Correspondence

A **one-to-one correspondence** is a function that operates between two sets where every element of one set is paired with exactly one element of the other set and every element of the other set is paired with exactly one element of the first set.This kind of function is also called a **bijection** or **bijective function**.

DEFINITION: Empty Set

The **empty set** is the set which contains no elements. it is often denoted $\{\}$ or $\emptyset$.

DEFINITION: Power Set

The **power set** of a set is the set of *all possible* subsets of a set. Note, the empty set is a subset of every set. If the given set is denoted A, then $\mathcal{P}(A)$ is the notation for the power set of A.

DEFINITION: Cardinality

The **cardinality** of a (finite) set is the number of elements in that set. The cardinality of the empty set is zero. The notation for cardinality of a set S is $|S|$. So, $|\{\}| = 0$.

Comparing Finite Sets

We can determine the cardinality of a finite set by placing it into a one-to-one correspondence with a subset of the natural numbers. This is a process that you and I would also call counting!

Exercise

What is the cardinality of the following sets?

(a) $\{-1, 0, 1, 2, 3, 4, 5, 6, 7\}$

(b) $\{\heartsuit, \clubsuit, \spadesuit, \diamondsuit\}$

(c) { one, two, four, six, seven, nine, ten, eleven}

(d) $\{\ \{-1, 0, 1, 2\}\ ,\ \{\text{apple, orange, grape }\}, \diamondsuit, \heartsuit, \square, \triangle\}$

(e) $\{\ \{-1, 0, 1, 2\}\ ,\ \{\text{apple, orange, grape }\},\ \{\diamondsuit, \heartsuit, \square, \triangle\}\}$

The Power Set: The Set Of All Subsets

Defining the the power set as the set of all possible subsets of a set produces an interesting result about the cardinality of the power set of sets which all have the same cardinality.

GroupWork

Let's find the power sets $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ of the following sets (and their cardinality).

(a) A=$\{1, 2, 3, 4\}$

(b) B=$\{\square, \bigcirc, \triangle\}$

(c) C={ {a, b, c }, { d, e }}

(d) D=$\{\clubsuit\}$

(e) E={}

$\boxed{\texttt{THEOREM: } \textbf{Cardinality of the Power Set}}$

The cardinality of the power set $\mathcal{P}(A)$ is equal to $2^{|A|}$. In other words,

$$|\mathcal{P}(A)| = 2^{|A|}$$

Comparing Infinite Sets: Galileo's Paradox

Which of the following sets $\mathcal{A}$ or $\mathcal{B}$ has more elements?

$$\begin{aligned}\mathcal{A} &= \{0,\ 1,\ 2,\ 3,\ 4,\ \ldots = \text{the set containing all Natural Numbers}\\ \mathcal{B} &= \{0,\ 1,\ 4,\ 9,\ 16,\ \ldots = \text{the set containing the squares of all Natural Numbers}\end{aligned}$$

It is clear that we can place both $\mathcal{A}$ and $\mathcal{B}$ into a 1-to1 correspondence with each other so that they must have the same number of elements!
Galileo Galilei noticed this issue with infinite sets and so the problem of how to compare infinite sets (especially the two sets listed above) is known as **Galileo's Paradox**. Galileo believed that the concept of "equal," "greater than" or "less than" could not be applied to infinite sets precisely for this reason. He put it this way:

> So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Countable versus Uncountable

If a set of numbers can be put into a one-to-one correspondence with Natural numbers than it is said to be **countable**. The question is, are there sets of numbers which can NOT be put into a one-to-one correspondence. Such sets are called **uncountable**. Russian mathematician **Georg Cantor** (1845-1918) was one of the inventors of set theory and is famous for proving that the set of real numbers has a cardinality greater than the natural numbers, i.e. the real numbers are uncountable. In 1891 he published what is known as the diagonalization argument which can be shown to prove that the real numbers are uncountable.

Visualization of Cantor's Diagonalization Proof

$$\boxed{\begin{array}{ll}
s_1 & = 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\ldots\\
s_2 & = 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\ldots\\
s_3 & = 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\ldots\\
s_4 & = 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\ldots\\
s_5 & = 1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\ldots\\
s_6 & = 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0\ldots\\
s_7 & = 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\ldots\\
s_8 & = 0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\ldots\\
s_9 & = 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\ldots\\
s_{10} & = 1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,1\ldots\\
s_{11} & = 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0\ldots\\
\vdots & \quad \vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\ddots
\end{array}}$$

$$\boxed{s \ = 1\,0\,1\,1\,1\,0\,1\,0\,0\,1\,1\ldots}$$

DEFINITION: Aleph Null or $\aleph_0$

The cardinality of the set of the natural numbers is denoted $\aleph_0$ and said to be "aleph null" or "aleph zero." Sets with this cardinality are often said to be **countable** or **countably infinite** or **denumerable**. $\aleph_0$ is said to be the first of the **transfinite numbers**.

GroupWork

Consider the following dozen infinite sets comparisons. Write down $>$, $<$ or $=$ between each pair of sets to indicate their relative size. **HINT: Think about which of these sets can be compared to the real number line and which can be compared to the counting numbers.**

	SET #1	SET #2
1:	$\{1,2,3,4,\ldots\}$ [all natural numbers]	[The natural numbers starting with 3] $\{3,4,5,6,\ldots\}$
2:	$\{1,2,3,4,\ldots\}$ [all natural numbers]	[All even natural numbers] $\{2,4,6,8,\ldots\}$
3:	$\{1,2,3,4,\ldots\}$ [all natural numbers]	[All odd natural numbers] $\{1,3,5,7,\ldots\}$
4:	$\{1,2,3,4,\ldots\}$ [all natural numbers]	[All unit fractions] $\{1,1/2,1/3,1/4,\ldots\}$
5:	$\{1,2,3,4,\ldots\}$ [all natural numbers]	[All points on an infinite line]
6:	[All points on a finite line segment]	[All points on an infinite line]
7:	$\{1,3,5,7,\ldots\}$ [all odd natural numbers]	[All multiples of four] $\{4,8,12,16\ldots\}$
8:	$\{10,20,30,40,\ldots\}$ [all multiples of 10]	[All multiples of four] $\{4,8,12,16\ldots\}$
9:	[All points on an infinite line]	[All points on a line 1" long]
10:	[All points inside a unit circle]	[All points on the circumference of a unit circle]
11:	[All points inside a unit circle]	[All points inside a unit square]
12:	[All points on a line 1/2" long]	[All points on a line 1" long]

QUESTION: What is the cardinality of the power set of the natural numbers, i.e. $|\mathcal{P}(\mathbb{N})|$?

EXTRA CREDIT OPPORTUNITY: (10 POTD points) Write a 250-300 word short essay about your understanding of the term infinity and discuss your reaction to the idea that there are relative sizes of infinity as proved by the existence of uncountable sets. **Due Fri March 29.**