

Class 19: Monday March 25

The Fundamental Theorem of Calculus

`DEFINITION`: **The Fundamental Theorem of Calculus**

When $f(x)$ is a function that is "nice" (i.e. continuous) on $[a, b]$ and there exists an anti-derivative of $f(x)$ called $F(x)$ so that $F'(x) = f(x)$ the Fundamental Theorem of Calculus (FTC) says that

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

But this means that we can replace the $f(x)$ in the integral with $F'(x)$, so that

$$\int_a^b F'(x)dx = F(b) - F(a)$$

In other words the value of the definite integral of $f(x)$ from $x = a$ to $x = b$ is simply the value of the difference between the anti-derivative $F(x)$ evaluated at the end points, i.e. $F(b) - F(a)$.

Another way of saying this is

> The **integral** of the **derivative** of a function from a to b can be evaluated as the difference in the value of the function at b minus the value of the function at a.

This is basically saying that DIFFERENTIATION and INTEGRATION are inverse processes of each other.

EXAMPLE

Previously we showed that $\int_0^3 (x^2 + 1)\ dx = 12$ by using limits. Now we can do so again by using the FTC.

In order to do so, we need to find a function $F(x)$ whose derivative equals the function being integrated (called the **integrand**) in the given definite integral.

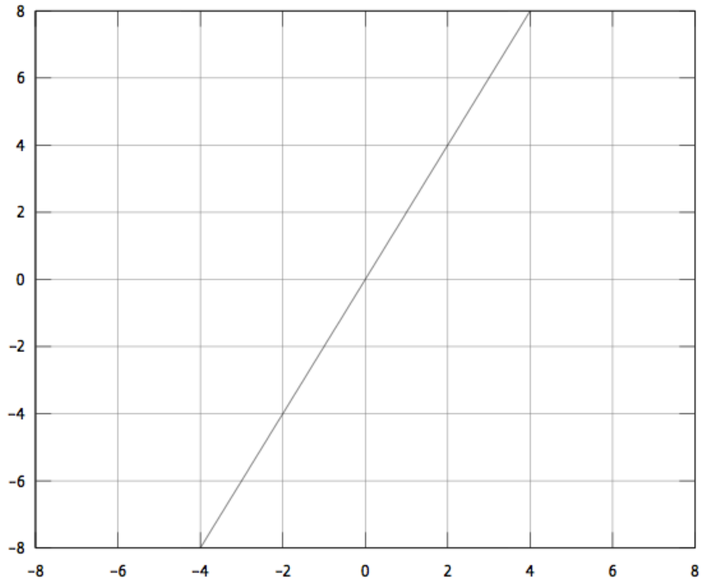
I claim that $F(x) = \frac{x^3}{3} + x$ is exactly the function that has this property. Using the FTC,

$$\begin{aligned} \int_0^3 (x^2 + 1)\ dx &= \int_0^3 F'(x)\ dx \\ &= F(3) - F(0) \\ &= \frac{3^3}{3} + 3 - (\frac{0^3}{3} + 0) \\ &= \frac{27}{3} + 3 - (0 + 0) \\ &= 9 + 3 \\ &= 12 \end{aligned}$$

Applying the Fundamental Theorem of Calculus

Consider the function $y = f(t) = 2t$ sketched on the interval $[0, x]$.

The (signed) area under this graph from $t = 0$ to $t = x$ is given by $\displaystyle\int_0^x 2t\ dt$.



Use geometry to find the area trapped by the graph of the function $f(x) = 2x$, the x-axis and the lines $x = 0$ and $x = b$.

GROUPWORK

In groups of 3 or 4 try to fill out the table below. Not everyone has to do every calculation. Split up the work among the members of the group.

For example, when $b = -1$, use geometry to find the area trapped by the graph of the function $f(x) = 2x$, the x-axis and the lines $x = 0$ and $x = -1$. (What shape is this?)
AREA=_____________

For example, when $b = 1$, use geometry to find the area trapped by the graph of the function $f(x) = 2x$, the x-axis and the lines $x = 0$ and $x = 1$. (What shape is this?)
AREA=_____________

For example, when $b = 2$, use geometry to find the area trapped by the graph of the function $f(x) = 2x$, the x-axis and the lines $x = 0$ and $x = 2$. (What shape is this?)
AREA=_____________

For example, when $b = 3$, use geometry to find the area trapped by the graph of the function $f(x) = 2x$, the x-axis and the lines $x = 0$ and $x = 3$. (What shape is this?)
AREA=_____________

For example, when $b = -2$, use geometry to find the area trapped by the graph of the function $f(x) = 2x$, the x-axis and the lines $x = 0$ and $x = -2$. (What shape is this?)
AREA=_____________

For example, when $b = -3$, use geometry to find the area trapped by the graph of the function $f(x) = 2x$, the x-axis and the lines $x = 0$ and $x = -3$. (What shape is this?)
AREA=_____________

The end-point value b	Area from $x = 0$ to $x = b$
$b = 1$	
$b = 2$	
$b = 3$	
$b = 4$	
$b = -3$	
$b = -2$	
$b = -1$	
$b = 0$	

Can we figure out what the pattern (i.e. function of b) that can be used to determine the value of $\int_0^b 2x\ dx$ as b changes?

The Fundamental Theorem of Calculus says that we can evaluate $\int_0^x 2t\ dt$ if we can find a function $F(t)$ such that $F'(t) = 2t$. But we have previously shown that when $y = x^2$ its derivative is the function $y' = 2x$. So this means that the function $F(t) = t^2$ has the derivative $F'(t) = 2t$.

Therefore, we can use the FTC to evaluate (and simplify) $\int_0^x 2t\ dt =$ ________________________.

DIFFERENTIATION and INTEGRATION ARE INVERSE PROCESSES OF EACH OTHER!
ANTI-DIFFERENTIATION IS INTEGRATION!