Mathematics As A Liberal Art

Math 105 Spring 2024 Fowler 309 MWF 3:00pm- 3:55pm

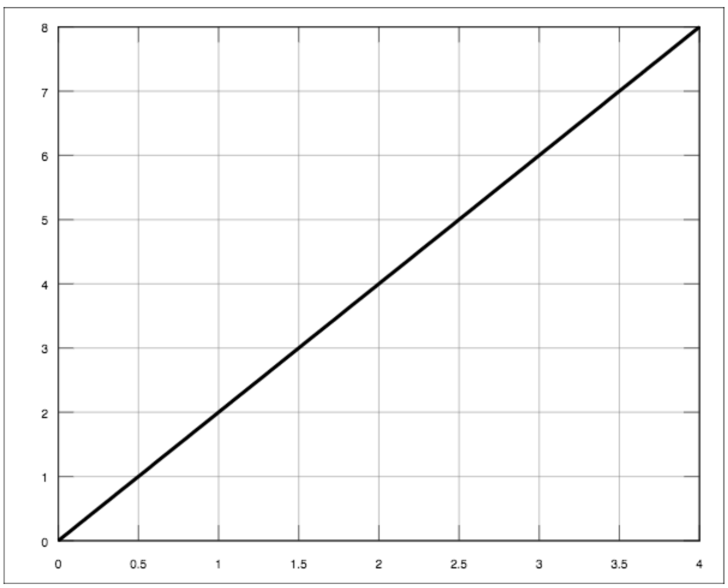
 http://sites.oxy.edu/ron/math/105/24/

Class 18: **Friday March 22**

Infinitesimals (continued): The Integral Calculus

Now that we have explored the Differential Calculus we will turn our attention to the Integral Calculus.
Integration has many uses, but we will put it in the context of finding the exact "area under a curve."

Suppose we consider the function $y = f(x) = 2x$. I have included the graph this function below on the interval from x = 0 to x = 4.



Exercise

Use geometry to find the area shaded below (trapped by the graph of the function, the x-axis and the lines $x = 0$ and $x = 4$).
We call this area

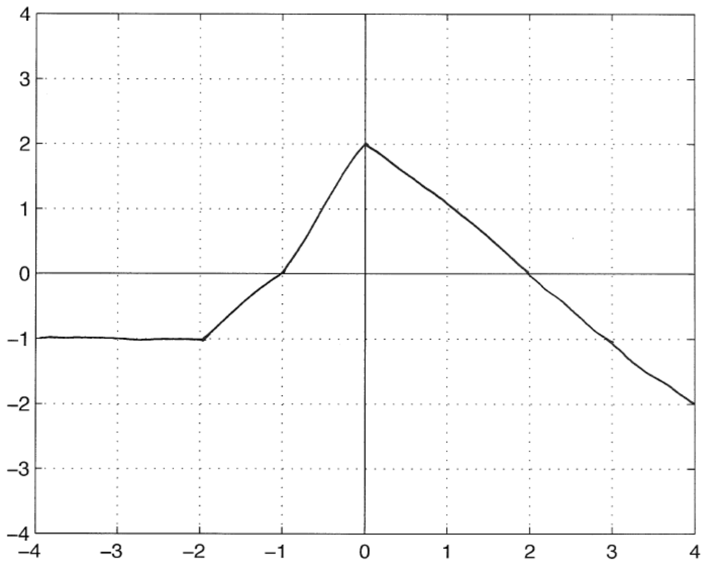
$$\int_0^4 (2x)\ dx$$

These symbols are read as "the integral from 0 to 4 of the function $f(x) = 2x$ with respect to x" and has the value of the area that we sketched in the figure above.

`DEFINITION`: **The Definite Integral**

The number represented by $\int_a^b\ f(x)\ dx$ is known as the definite integral of $f(x)$ from $x = a$ to $x = b$. It represents the value of the signed area trapped between the function $f(x)$ and the x-axis where area "below" the y-axis is considered to be negative.

Estimating Integrals Using Geometry



GROUPWORK

Use the figure above to evaluate the following integrals.

$$\int_{-1}^{4} f(x)\ dx$$

$$\int_{0}^{2} f(x)\ dx$$

$$\int_{-4}^{-2} f(x)\ dx$$

THEOREM: **Properties of The Definite Integral**

$$\begin{aligned}
\int_a^a f(x)\ dx &= 0 \\
\int_a^b f(x)\ dx &= -\int_b^a f(x)\ dx \\
\int_a^b [f(x)+g(x)]\ dx &= \int_a^b f(x)\ dx + \int_a^b g(x)\ dx \\
\int_a^b [f(x)-g(x)]\ dx &= \int_a^b f(x)\ dx - \int_a^b g(x)\ dx \\
\int_a^b c\cdot f(x)\ dx &= c\int_a^b f(x)\ dx \\
\int_a^c f(x)\ dx + \int_c^b f(x)\ dx &= \int_a^b f(x)\ dx \\
\int_a^b f(x)\ dx &= -\int_b^a f(x)\ dx
\end{aligned}$$

The Quadrature Problem

The problem of determining the area of irregular shapes is known as the **quadrature** problem. This is related to the class problem of "Squaring the Circle." This was the idea that one could come up with a square that had the exact same area as a circle. Eventually it was shown to be impossible. (This is true because the area of a unit circle is exactly π and in order for a square to have the same area each side would have to equal $\sqrt{\pi}$, which is not only **irrational** but also **transcendental**!)

One of the key benefits of the Integral Calculus is that it is a method for computing various kinds of quadrature problems. Consider the area represented by the area under the curve of $y = x^2 + 1$, above the x-axis and between the lines $x = 0$ and $x = 3$. We know we can represent this area as the definite integral $\mathcal{I} = \int_0^3 (x^2 + 1)\, dx$. But how can we determine its value? Let's estimate it!
Our first estimate is found by considering the function to be constant over the entire interval [0,3] and finding the area of the corresponding rectangle. We could pick any point to use for our height of the rectangle but let's choose the right-most point.

Estimate the value by breaking the interval [0,3] into 2 pieces of equal size.

Estimate the value by breaking the interval [0,3] into 3 pieces of equal size.

Find another estimate by breaking up the interval [0,3] into 6 pieces of equal size.

Which of these answers is the most accurate? What would we have to do to get the best possible estimate (i.e. exact) area represented by the definite integral?

I have made a table of estimates to the area in question using an increasing number of pieces into which the interval [0,3] is divided.

Number of Pieces	Estimate of Area
1	
2	
3	
6	
12	13.256250000000
24	12.570312500000
48	12.283203125000
1,000	12.013504500000
5,000	12.002700180000
10,000	12.001350045000
100,000	12.000135000450
1,000,000	12.000013500005
$\vdots$	$\vdots$
∞	

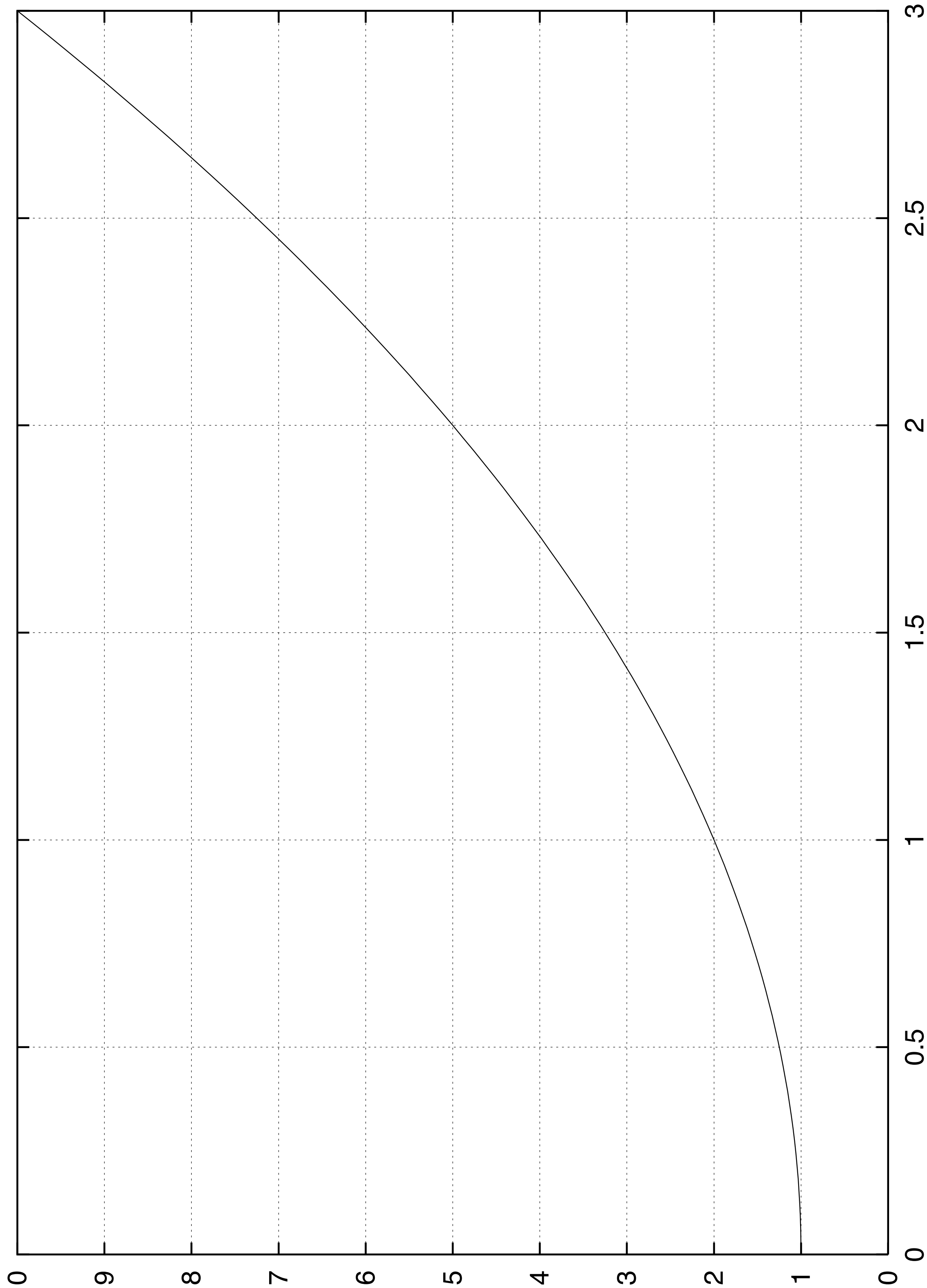
Therefore we can say that the exact value of $\int_0^3 (x^2+1)\ dx = 12$

Infinity, Infinitesimals and Calculus

We said that doing Calculus would involve infinity and infinitesimals. Have we seen that so far? What is the infinitesimal quantity involved in evaluating the definite integral? What was the infinitesimal quantity involved in evaluating derivatives? How was infinity involved in these Calculations?

`DEFINITION`: **infinitesimal**

An **infinitesimal** quantity is one that is indefinitely small, that is it is approaching (but never equals!) zero.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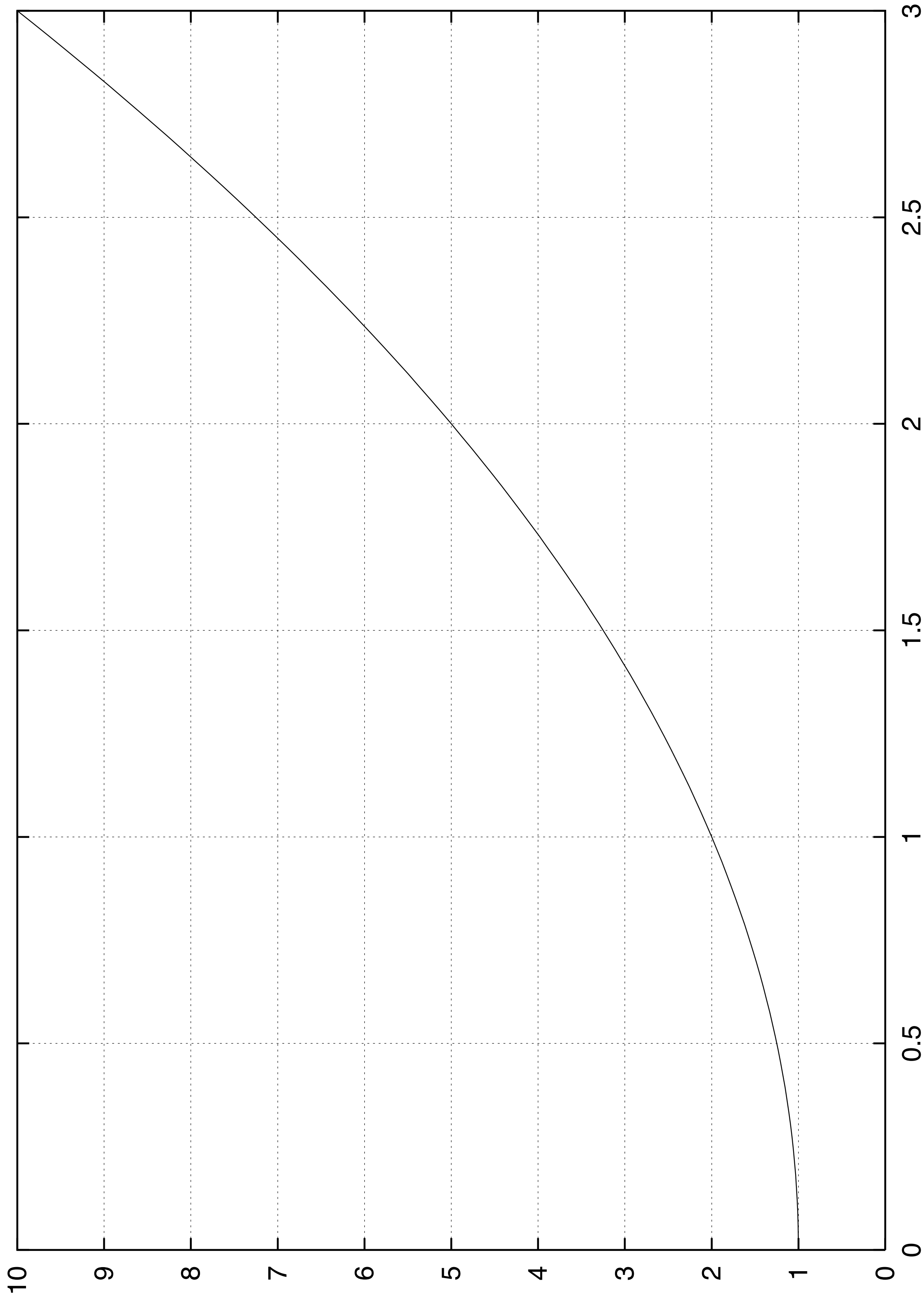
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