Class 17: Wednesday March 20

Introduction To Infinitesimals and the Differential Calculus

DEFINITION: **Slope Of A Line**

The **slope** of a line is the ratio of the change in the output (y-change) over the change in the input (x-change). It is usually depicted as

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{RISE}}{\text{RUN}}$$

So if a line has the points $(x_1, y_1$ and (x_2, y_2) one can compute the slope using the formula above.

EXAMPLE

What's the slope of the line that does through the point $(-2, 5)$ and $(1, 2)$? Sketch it below.

NOTE

A line with POSITIVE slope will have segments in the FIRST and THIRD QUADRANTS. A line with **NEGATIVE** slope will have segments in the SECOND and FOURTH QUADRANTS.

GroupWork

Sketch the lines and find the slope of each line

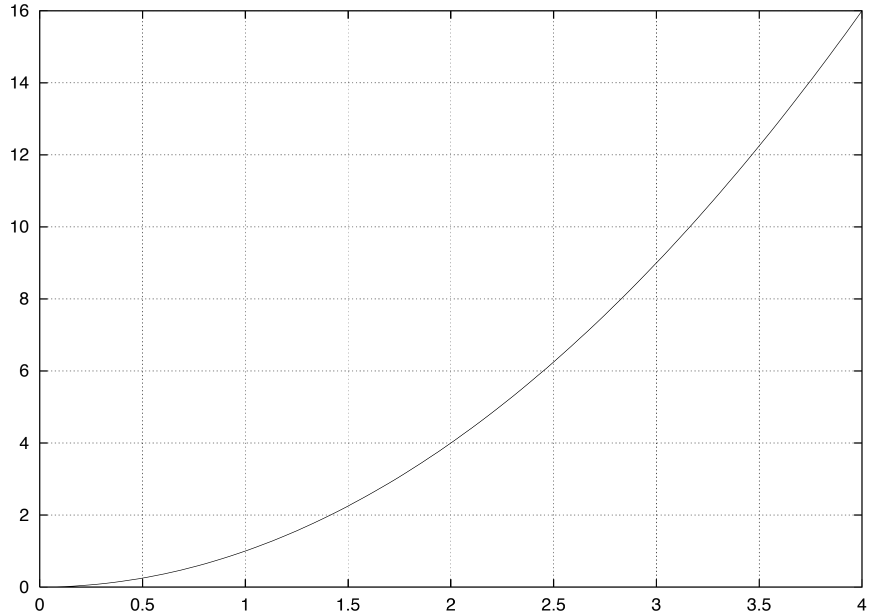
(a) the line that goes through the points (0,0) and (-8,8)

(b) the line that goes through the points (-3,4) and (2,4)

(c) the line that goes through the points (-2,-4) and (3,3)

(d) the line that goes through the points (-2,-4) and (-2,5)

Finding The Slope of Nonlinear Graphs



EXAMPLE

Consider the graph of the parabola given above. We would like to find the "slope" of the graph at the point (2,4). A parabola is not like a line. It does not "go straight." Its "slope" is always changing. Our definition of "slope" for such a graph will have to take this into account.

Connect the points (2,4) and (4,16). Find the slope of this line.
This will be an approximation to the slope of the parabola at the point (2,4).

Connect the points (2,4) and (3,9). Find the slope of this line.

Connect the points (2,4) and (2.5, 6.25). Find the slope of this line.

Find the slope of the line connecting (2,4) and (2.1,____________)

Find the slope of the line connecting (2,4) and (2.001,____________)

Do you see a pattern forming? What is it?

GroupWork

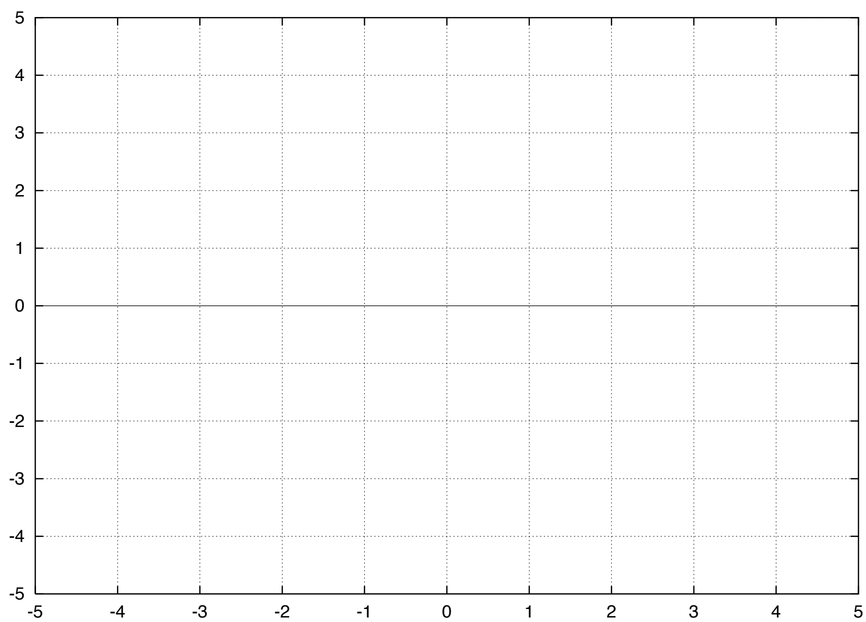
In groups, you will compute the slope of the graph $y = x^2$ at one of several different points. Write your group's answer here:

We now have the slope of the graph to $y = x^2$ at many different points–(-2,4), (-1,1), (1/2,1/4), (1,1), (2,4) and (3,9).
We will use this information from our groups to graph a new function.
We will plot points corresponding to the x-values -2, -1, 1/2,1, 2 and 3.

The y-values will be the slopes we just calculated corresponding to these x-values.

when x = -2, the corresponding slope $y =$
when x = -1, the corresponding slope $y =$
when $x = 1/2$, the corresponding slope $y =$
when $x = 1$, the corresponding slope $y =$
when $x = 2$, the corresponding slope $y =$
when $x = 3$, the corresponding slope $y =$

Do you see a pattern yet between x and y? Plot the points below on the given axes (below)



What do you think is going on? What is the relationship between x and y depicted in the graph?
If you "connect the dots" what kind of graph do you get?
Can you write the equation that describes this graph?

Explaining The Concept

What we have been doing is the equivalent of looking at the slope between two points at (a, a^2) and $(a+h, (a+h)^2)$ and seeing if we see a pattern when h gets smaller and smaller. This process is known as **taking the limit** and is usually written $\lim\limits_{h \to 0}$.

The value of h is said to be **infinitesimal**, i.e. really really really small (BUT NOT ZERO!)

`DEFINITION`: **infinitesimal**

An **infinitesimal** quantity is one that is indefinitely small, that is it is approaching (but never equals!) zero.

Differential Calculus: Using Infinitesimals

`DEFINITION`: **Derivative**

The derivative $f'(x)$ of a function $y = f(x)$ at the point $x = a$ is given by finding the value of the following limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and then plugging in $x = a$ into the derivative function $f'(x)$.
One can also plug $x = a$ into the definition first and look at the definition this way:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

`NOTATION`: **Derivative**

The derivative of the function $y = f(x)$ can be written as $\frac{dy}{dx}$ (Leibniz's notation) where

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\text{CHANGE IN INPUT} \to 0} \frac{\text{Change in Output}}{\text{Change in Input}}$$

EXAMPLE

If we consider the function $f(x) = x^2$ we can show that $f'(x) = 2x$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2hx + h^2}{h} \\
&= \lim_{h \to 0} 2x + h \\
f'(x) &= 2x
\end{aligned}$$

This above calculation is an example of the kind of process that **Sir Isaac Newton** (1643-1727) and **Gottfried Wilhelm Leibniz** (1646-1716) went through in the late 17th Century which later became known as **The Calculus**.

Exercise

Consider the equation of a line of the form $y = f(x) = mx + b$. Use the definition of the derivative as shown in the previous example to obtain the important result that
"The derivative of the equation of a line is exactly equal to its slope".