Class 16: Monday March 18

Introduction To Infinite Series and Sequences

DEFINITION: **Infinite Sequence**

A sequence is an ordered list of numbers generated by a function which takes the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ as input and outputs one thing for each input. This output set is called a sequence is often denoted:

$$a_0, \quad a_1, \quad a_2, \quad a_3, \quad \ldots \qquad \text{or } \{a_k\}_{k=0}^{\infty}$$

The **terms** or "elements" of the sequence are the numbers a_k where k is a natural number. The function $a(k)$ tells you how to find the kth element of the sequence. If the sequence gets closer and closer to a particular answer as the index of the terms grows higher and higher this answer (which can be thought of as the last term in the sequence) is called the **limit** of the sequence. If the limit $L = \lim\limits_{k \to \infty} a(k)$ of the sequence exists (and is finite) the sequence is acid to CONVERGE. Otherwise, it is a **divergent** sequence.

EXAMPLE

Consider the sequence given by $a_k = \left(\dfrac{1}{2}\right)^k$. Let's write down the first three or four terms of the sequence and see if we can determine the limit of the sequence.

Exercise

What's the next two terms in the following sequences? Can you find a formula for the pattern or function predicting the kth element?

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

(b) 2, 4, 8, 16, ...

(c) 1, 3, 5, 7, ...

(d) 1, -1, 1, -1, ...

GROUPWORK

What is the limit of each of the sequences you analyzed above? Classify each sequence as either convergent or divergent.
(a)
(b)
(c)
(d)

QUESTION **Can a sum of an infinite list of positive numbers be finite?**

Summing an Infinite List of Numbers.

If someone gives you a list of numbers, even a long list of numbers (like 1000 of them), it is at least theoretically possible for you to use your calculator or computer to find the sum total of this list. Now, suppose someone gives you an *infinite* list of numbers, for example, the sequence $\left\{\dfrac{1}{k^k}\right\}_{k=1}^{\infty}$:

$$1, \qquad 1/2^2, \qquad 1/3^3, \qquad 1/4^4, \qquad 1/5^5, \qquad 1/6^6, \qquad etc.$$

Is it possible to find the total? What could "find the total" mean if you are adding up an infinite list of numbers?

$\boxed{\text{GROUPWORK}}$

In small groups use your calculators to begin with the first number on the infinite list above, 1, and progressively add each successive number on the list, keeping track of the subtotals you get by placing them in the chart below, with seven places after the decimal.

n	n^{th} subtotal	
1	1	$= 1.0000000\ldots$
2	$1 + 1/2^2$	$=$
3	3rd subtotal	$=$
4	4th subtotal	$=$
5	5th subtotal	$=$
6	6th subtotal	$=$

What do you find happening to the subtotals? If this trend continues, what will be the first four digits of all the subtotals beyond those in the table? None of the numbers in the list, beyond a certain point, seem to be affecting the first four digits of the subtotals. So, if you were somehow able to add up *all* of the numbers in the infinite list, what do you think the first four digits of the total would be?

Find the first six decimals of the sum of the numbers in our infinite list.

What would you do to find the first ten decimals of the sum of the numbers in our infinite list? (You don't have to actually do it.)

How would you describe the sum of our infinite list of numbers using the concept of "limit"?

DEFINITION: **Infinite Series**

An **infinite series** (or oftentimes just "series") is the sum of the terms of a given infinite sequence.

Formal Language of Infinite Series.

Using the proper terminology, we will discuss what you have just done. We had a list of numbers (which we know is called a *sequence* of numbers):

$$1, \qquad 1/2^2, \qquad 1/3^3, \qquad 1/4^4, \qquad 1/5^5, \qquad 1/6^6, \qquad \text{etc.}$$

which we call the **TERMS** of the **INFINITE SERIES** and denote it using the Greek letter $\sum$

$$1 + 1/2^2 + 1/3^3 + 1/4^4 + 1/5^5 + 1/6^6 + \ldots = \sum_{k=1}^{\infty} 1/k^k.$$

We tried to find the sum of this infinite series by looking at its **SEQUENCE OF PARTIAL SUMS** (list of subtotals):

$$
\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + 1/2^2 \\
S_3 &= 1 + 1/2^2 + 1/3^3 \\
S_4 &= 1 + 1/2^2 + 1/3^3 + 1/4^4 \\
&\vdots \\
S_n &= 1 + 1/2^2 + 1/3^3 + 1/4^4 + \ldots + 1/n^n \\
&\vdots
\end{aligned}
$$

We found that the sequence of partial sums S_n seemed to have a **LIMIT** (i.e. the subtotals were stabilizing to a particular value), and that the limit of this sequence of partial sums was the **SUM** of the infinite series:

$$\sum_{k=1}^{\infty} 1/k^k = \lim_{n \to \infty} S_n.$$

When the sequence of partial sums S_n of an infinite series has a limit, the infinite series is said to **CONVERGE**.

When the sequence of partial sums S_n do not have a limit, the infinite series is said to **DIVERGE**.

Therefore, in this case, the infinite series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^k}$ that we have been examining

CONVERGES.

THEOREM

If the series $\displaystyle\sum_{k=1}^{\infty} a_k$ is convergent, then $\displaystyle\lim_{k \to \infty} a_k = 0$

We Have A Test For Divergence!

Note the contrapositive statement of this theorem is also rather important to remember:

$$\text{IF } \lim_{k \to \infty} a_k \neq 0, \text{ THEN } \sum_{k=1}^{\infty} a_k \text{ is } \textbf{divergent}$$

This is sometimes called the **Divergence Test for Infinite Series**

Finding The Sum Of An Infinite Series

DEFINITION: **Geometric Series**

A **geometric series** is a special type of infinite series where there is a fixed ratio between successive terms in the series. A geometric series is often written

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \dots$$

the first term of the series is a and the fixed ratio is r.

Geometric series are very lovely because there is a simple formula to find out not only whether they converge or diverge, but if they converge, we can know exactly what NUMBER they converge to. Being able to tell what an infinite sequence of terms adds up to is pretty special!

THEOREM

The sum of an infinite geometric series that starts with a and has a fixed ratio r is often depicted S_∞ and computed as

$$S_\infty = \frac{a}{1-r}.$$

The geometric series converges when $|r| < 1$ and diverges when $|r| \geq 1$.

PROOF

Exercise

Determine whether the following series are geometric, and if so, find their sum (if the series is convergent).

(a) $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \dfrac{1}{16} \dots$

(b) $2 + 4 + 8 + 16 + \dots$

(c) $3 + 0.3 + 0.03 + 0.003 + 0.0003 + \dots$

(d) $1 + (-1) + 1 + (-1) + \dots$