Mathematics As A Liberal Art

Math 105 Spring 2024 — Fowler 309 MWF 3:00pm- 3:55pm

 — http://sites.oxy.edu/ron/math/105/24/

Class 13: Wednesday February 28

Mathematical Logic Continues: Implications

THEOREM: **DeMorgan's Law**

DeMorgan's Law allows us to transform statements involving negations and conjunctions into statements involving negations and disjunctions and *vice versa*. Specifically, DeMorgan's Law states:

> The negation of a conjunction is the disjunction of the negations AND the negation of a disjunction is the conjunction of the negations.

These are actually two different statements (sometimes called DeMorgan's Laws). They are best expressed mathematically as

$$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$$

and

$$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$$

By considering these logical statements as Venn Diagrams it can be shown that DeMorgan's Laws are equivalent to the following results:

$$\overline{A \cup B} \equiv \overline{A} \cap \overline{B} \text{ and } \overline{A \cap B} \equiv \overline{A} \cup \overline{B}$$

EXAMPLE

Let's draw pictures representing DeMorgan's Laws using Venn Diagrams.

GroupWork

Confirm DeMorgan's Laws By Using Truth Tables.

Logical Implication: *modus ponens* and *modus tollens*

The oldest and most famous logical implications are known as *modus ponens* (Latin for "the way that affirms by affirming") and *modus tollens* (Latin for "the way that by affirming, denies"), often abbreviated as MP and MT, respectively.

Mathematically, MP can be represented as $[(p \to q) \wedge p] \to q$ while MT can be represented as $[(p \to q) \wedge \neg q] \to \neg p$

Let's verify these statements are valid arguments by using truth tables.

Modus Ponens

p	q	
T	T	
T	F	
F	T	
F	F	

Modus Tollens

p	q	
T	T	
T	F	
F	T	
F	F	

From Boolean Algebra To Logical Argument

The real power of these logical implications occurs when one thinks of the Boolean variables p and q as propositions, i.e. actual English statements that are either truth or false.
For example,

$p =$ "The sun is shining." $\qquad q =$ "You will get a sunburn."

Modus ponens $[(p \to q) \wedge p] \to q$ is then the equivalent of the English statements that make the following argument:

IF the sun is shining THEN you will get a sunburn.
The Sun is Shining.

CONCLUSION: You will get a sunburn.

Modus tollens $[(p \to q) \wedge \neg q] \to \neg p$ is then the equivalent of the English statements that make the following argument:

Exercise

Use your own English sentences and write down examples of valid logical arguments using MP and MT.

QUESTION Do all your arguments "make sense"? If they don't, that's the difference between valid and sound arguments.

Validity Versus Soundness

DEFINITION: **Valid Argument**

A **valid argument** is an argument where the conclusion is a logical result of its premise(s). In other words, the conclusion is a consequence of the premise(s).

Usually, this means that the premises are true and lead to a true conclusion, but you can still have a valid argument where the premises and conclusions are both false.

DEFINITION: **Sound Argument**

A **sound argument** is an argument where the argument is **valid** and the premise(s) of the argument is true.

GroupWork

Use English sentences to write down examples of logical arguments that are both valid and sound using MP and MT.

QUESTION Can you have a sound argument that is not valid? How about a valid argument that is not sound?