

Class 12: Monday February 26

Introduction to Mathematical Logic

A Brief History of Mathematical Logic

The question of how to systematize the process by which statements can be strung together to produce mathematical proof was first popularized by the Greek philosopher Aristotle. **Aristotlean logic** and the use of syllogisms (subject-predicate propositions) held sway for thousands of years until a British mathematician, named George Boole (1815-1864), came up with a way to describe the language used in logical proofs using symbols that could be manipulated algebraically. The result was something called **Boolean algebra.** Other important contributions to mathematical logic were made by Bertrand Russell and Kurt Gödel.

`DEFINITION`: **Boolean variable**

A **Boolean variable** or simply "Boolean" is a variable which can be either true or false (sometimes depicted as 1 or 0).

`DEFINITION`: **proposition**

A **proposition** or is a declarative statement (usually a sentence) that is either true or false but not both. It is often represented by a Boolean variable.

`DEFINITION`: **Truth table**

Truth tables are a visual way of analyzing logical propositions that was invented by Ludwig Wittgenstein (1889-1951). They depict the final value of a statement or logical operation involving two or more Boolean variables (a variable which can take on the value **T** (True) or **F** (False)).

p	q	$f(p,q)$
T	T	T or F
T	F	T or F
F	T	T or F
F	F	T or F

Logical Operators: AND, OR and NOT

The first logical operator we will learn is the **negation** operation. This has the result of swapping a Boolean variable's value. In other words $T \to F$ and $F \to T$. It is the only **unary** operation (it only has one input and produces one output).

Negation

The truth table for the **NOT** operator (which is usually denoted as $\neg$) is given below:

p	$\neg p$
T	F
F	T

Logical Conjunction (AND)

The truth table for the **AND** operator (which is usually denoted as $\wedge$ and is often called a **logical conjunction**) is given below. The definition of the **AND** operation is that it is only TRUE if both of its inputs is true. (This implies that it is FALSE when at least one of its inputs is false.).

The logical conjunction in English can be read as "**The statement is true only if p and q are both true.**"

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical Disjunction (OR)

The truth table for the **OR** operator (which is usually denoted as $\vee$ and is often called a **logical disjunction**) is given below. The definition of the **OR** operation is that it is only TRUE when at least one of its inputs is true. (This implies that the **OR** output is only FALSE when both inputs are FALSE.)

The logical disjunction in English can be read as "**The statement is true if either p or q is true.**"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical Conditional (IF , THEN)

The most important of the binary logical operators is the one known as $p \to q$, which represents logical statements which can be read "If p, then q." Almost all mathematical theorems are written in this format, also known as the **material implication** binary operator.

In English, one can interpret the logical conditional as $p \to q$ as meaning that "**If p is true, then q is true.**"
The convention that follows is that $p \to q$ is FALSE only when p is true and q is false.
It's Truth Table is:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The best way to think of the material implication is as the statement "Either p is not true, OR q is true." In symbols this is $\neg p \vee q$. We can show that this given statement has the same truth table as $p \to q$. In that case we say that the two statements are **logically equivalent**.

DEFINITION: **logically equivalent**

Two statements are said to be logically equivalent if their truth tables are identical.

DEFINITION: **Tautology**

A **tautology** is a statement that is always true. An example is $p \vee \neg p$.

EXAMPLE

Let's show that $\neg p \vee q$ is logically equivalent to $p \to q$.

p	q	$\neg p$	$\neg p \vee q$
T	T		
T	F		
F	T		
F	F		

GROUPWORK

Compute truth tables for the following statements

The **Contrapositive** $\neg q \to \neg p$.

The **Converse**: $q \to p$

The **Inverse**: $\neg p \to \neg q$

Summary of Logically Equivalences

The contrapositive is logically equivalent to the conditional $(\neg q \to \neg p \equiv p \to q)$

The converse is logically equivalent to the inverse. $(q \to p \equiv \neg p \to \neg q)$

Representing Binary Operators Visually: Venn Diagrams

John Venn (1834-1923) invented a technique for representing logical binary operations, which he referred to as "Eulerian circles" (after the great Swiss mathematician) Leonhard Euler (1707-1783). Euler is perhaps best known for the most famous equation in all of mathematics, called "Euler's identity" or Euler's equation.

$$e^{i\pi} + 1 = 0$$

It contains some of the most important constants in mathematics: e the base of the natural logarithms, i, the imaginary unit representing $\sqrt{-1}$, i.e. $i^2 = -1$, π the ratio of the circle's circumference to its diameter and the binary digits 1 and 0.

INTERSECTION is equivalent to CONJUNCTION

A Venn diagram is a graphical illustration of the relationships between two sets A and B. For example, the **logical conjunction** $A \wedge B$ also known as the **AND** operator are the same things as $A \cap B$ or A intersection B, depicted below.

UNION is equivalent to DISJUNCTION

The **logical disjunction** $A \vee B$ also known as the **OR** operator is the same thing as $A \cup B$ or A union B, depicted below.

$\boxed{\textbf{Exercise}}$

Can you draw pictures of Venn Diagrams which represent $p \to q$ and $q \to p$? Label these as the converse, conditional, contrapositive and inverse, appropriately.

HINT: Remember $p \to q$ is $\neg p \vee q$!