Mathematics As A Liberal Art

Math 105 Spring 2024 Fowler 309 MWF 3:00pm- 3:55pm

 http://sites.oxy.edu/ron/math/105/24/

Class 9: Wednesday February 14

All Numbers Are Not Rational!

The Rational Numbers

The Rational Numbers, the set of all possible numbers formed by the ratio of two integers is represented by the symbol $\mathbb{Q}$.

QUESTION

Can you find a number between 1001/1003 and 1002/1003 that is the ratio of two natural numbers? **If possible, write down at least one such number, as well as the process for finding it, in the space below.**

ANSWER________________________

QUESTION

Can you find a number between *any* two rational numbers that is also a rational number?

ANSWER________________________

QUESTION

Can every number imaginable be written as a ratio of two rational numbers?

ANSWER________________________

THEOREM

$\sqrt{2}$ can not be written as the ratio of two natural numbers, i.e. $\sqrt{2}$ is irrational.

PROOF

Assume that a and b are two natural numbers that are **relatively prime** (i.e. they have no whole number factors that are in common to each other). We shall prove that $\sqrt{2}$ is irrational by using the technique of **proof by contradiction**.

In **Proof by Contradiction** you assume the opposite of the statement you are trying to prove and show that if you make logical mathematical operations you will arrive at a contradiction. This contradiction means your original assumption is false, so you have proved what you wanted!

STEP 1. Write down the opposite of what you want to prove: $\sqrt{2}$ is rational.

$$\begin{aligned} \text{Let } \sqrt{2} &= \frac{a}{b} && \text{where } a \text{ and } b \text{ have no common factors} \\ b\sqrt{2} &= a && \text{(Square both sides)} \\ b^2 2 &= a^2 && \text{This equation means that } a^2 \text{ must be even!)} \end{aligned}$$

STEP 2: Find a contradiction to demonstrate you have proved the opposite of the statement in STEP 1. We will show that our assumption that a and b have no common factors is FALSE.

Since $a^2 = 2b^2$ this means that a^2 is an even number, in other words it is a multiple of 2.

LEMMA

The product of any two even numbers is **even**.
The product of any two odd number is **odd**
The product of any odd number and any even number is **even**.

Since a^2 is a times a this means that a is an even number. Which means that we could write $a = 2c$ where c is any natural number.

$$\begin{aligned} b^2 2 &= a^2 && \text{(Let } a = 2c) \\ 2b^2 &= (2c)^2 && \text{(Square both sides)} \\ 2b^2 &= 4c^2 && \text{(Divide both sides by 2)} \\ b^2 &= 2c^2 && \text{This equation means that } b^2 \text{ must be even!)} \end{aligned}$$

Just like a, Since b^2 is b times b this means that b is an even number. **This is our contradiction!** Both a and b can not be even, or else they would have the common factor of 2 but we said they had no common factors. So, our assumption that $\sqrt{2}$ is rational is FALSE, so we have proved that the statement "$\sqrt{2}$ is rational" is FALSE, or that the statement "$\sqrt{2}$ is irrational" is true.

Pythagoras' Triumph (and Sorrow): The Length of the Diagonal Of the Square Is Irrational

Howard Eves, in *An Introduction to the History of Mathematics* (6th Edition, 1990) says on page 83:

> The discovery of the existence of irrational numbers was surprising and disturbing to the Pythagoreans. First of all, it seemed to deal a mortal blow to the Pythagorean philosophy that all depends upon the whole numbers. Next, it seemed contrary to common sense, for it was felt intuitively that any magnitude could be expressed by *some* rational number. The geometrical counterpart was equally startling, for who could doubt that for any two given line segments one is able to find a third line segment, perhaps very small, that can be marked off a whole number of times in each of the two given segments? But take as the two segments a side s and a diagonal d of a square. Now if there exists a third segment t that can be marked off a whole number of times into s and d , we would have $s = bt$ and $d = at$ where a and b are positive integers. But $d = s\sqrt{2}$, whence $at = bt\sqrt{2}$—that is, $a = b\sqrt{2}$ or $\sqrt{2} = a/b$, a rational number. Contrary to intuition, them there exists **incommensurable** line segments—that is line segments having no common unit of measure.

EXAMPLE

Let's prove that $\sqrt{6}$ is irrational.

GroupWork

What's easier, proving $\sqrt{3}$ is irrational or proving $\sqrt{8}$ is irrational? Prove the easier statement.

LEMMA

The product of any two even numbers is **even**.
The product of any two odd number is **odd**
The product of any odd number and any even number is **even**.

Exercise

Use the method of direct proof to prove the statements from the Lemma is true.