Class 8: Monday February 12

Prime Time: The Fundamental Theorem of Arithmetic

Revisiting The Primes

The natural numbers are the "counting" numbers, or all the whole numbers starting with 1. The set of all natural numbers $\{1, 2, 3, 4, \ldots\}$ is often denoted by the symbol $\mathbb{N}$. The field of mathematics devoted to studying the relationships between the natural numbers is known as **number theory.**

DEFINITION: prime number

A number is said to be **prime** if it is a positive integer greater than 1 without having any positive whole number divisors except for itself.

DEFINITION: composite number

A number is said to be **composite** if it is not a prime number. In other words, all numbers are either composite or prime.

DISCUSSION QUESTION **Are there more composite or prime numbers?**

The Density of the Primes

N	$\pi(N)$	$\pi(N)/N$
1,000	168	0.168
10,000	1,229	0.123
100,000	9,592	0.095
1,000,000	78,498	0.078
10,000,000	664,579	0.066
100,000,000	5,761,455	0.058

DEFINITION: prime density function

The number of prime numbers in the first N natural numbers is denoted by $\pi(N)$. The prime density function is usually written as $\pi(N)/N$.

GroupWork

What is $\pi(10)/10$? $\pi(20)/20$? $\pi(50)/50$? $\pi(100)/100$? How about $\pi(N)/N$ as $N \to \infty$?

The Fundamental Theorem of Arithmetic

THEOREM: **The Fundamental Theorem of Arithmetic**

Every natural number is prime or can be expressed as a product of primes that is unique apart from the order in which the product is written.

This is also sometimes called the **unique factorization theorem** or **the unique prime factorization theorem**.

This means that we can take a number and express it as the product of prime numbers.

EXAMPLE

We can show that

$$56 = 2 \times 2 \times 2 \times 7$$

We say that the prime factorization of 56 is $2^3 \times 7$.

Exercise

Find the prime factorization of 124. Is 124 perfect, abundant or deficient?

GroupWork

Find the prime factorization of 8128. Is 8128 perfect, abundant or deficient?

THEOREM: **Euler-Euclid Theorem**

Every even perfect number can be represented as $2^{n-1}(2^n - 1)$ where $2^n - 1$ is a prime number.

Revisiting The Primes

On Page 72 of *Discovering the Art of Mathematics: Number Theory* the following quote by American mathematician Don Zagier, Director of the Max Planck Institute for Mathematics, is given:

> **There are two facts about the distribution of prime numbers which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that despite their simple definition and role as the building blocks of the natural numbers, the prime numbers [...] grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behavior, and that they obey these laws with almost military precision.**

This quote articulates some of the motivation for why mathematicians spend so much time studying something as "simple" as the natural numbers.

DEFINITION: related primes

Mathematicians are so obsessed with studying primes that they have developed numerous terms to describe certain kinds of prime numbers as well as prime numbers that have relationships with each other.

twin primes are a pair of primes p and q such that $q - p = 2$.

cousin primes are a pair of primes p and q such that $q - p = 4$.

sexy primes are a pair of primes p and q such that $q - p = 6$.

Sophie Germain primes are a pair of primes such that p and $2p + 1$ are both prime.

Mersenne primes are prime numbers which have the form $2^p - 1$ where p is a prime number. There are only 48 known Mersenne prime numbers!

titanic prime is a prime number that has at least 1000 decimal digits. The smallest titanic prime is $10^{999} + 7$. (NOTE: a **googol** is the number 10^{100})

GroupWork

Find at least one example of each of the special prime numbers described above!

Number Theory: "The Queen of Mathematics

Number Theory is the subfield of mathematics devoted to the study of the integers (signed whole numbers).

The Handout ("**Important Theorems and Conjectures in Number Theory**") lists some of the important theorems and conjectures involving the prime numbers that we will be considering.

THEOREM

There are an infinite number of primes.

QUESTION **What does this theorem imply?**

There is no largest prime number.

There is some distribution of primes among the natural numbers.

CONJECTURE: Goldbach's Conjecture

Every even integer greater than 2 is the sum of two primes.

This is one of the oldest unsolved problems in all of mathematics. It has not been proven to be true, but it has been shown via computer to be true for all numbers up to approximately 4×10^{18} (i.e. four billion billion or four quintillion).

This form of the conjecture is sometimes known as **Goldbach's strong conjecture** to distinguish it from **Goldbach's weak conjecture** (sometime known as the three primes conjecture or the odd Goldbach conjecture) which is **Every odd integer greater than 5 is the sum of three primes**.

EXAMPLE

Show that 36 satisfies Goldbach's (strong) conjecture.

Exercise

Show that 100 satisfies Goldbach's strong conjecture.

Show that 35 satisfies Goldbach's weak conjecture.

Fermat's Last Theorem

For a very long time it was merely a conjecture that the following statement was only true for all integers x, y and z

$$x^n + y^n = z^n \quad \textbf{has no solutions for } n > 2$$

The proof of the above statement was finally given by Princeton Mathematics professor **Andrew Wiles** (1953–) in 1993-1994 and involves hundreds of pages of advanced mathematics.

Fermat's Last Theorem was considered one of the greatest unsolved mathematics problems of all time, having resisted a formal proof since its statement by Fermat in 1637. (How many years is that?!)

DEFINITION

An ordered triple of integers (x, y, z) is called a **Pythagorean Triple** if $x^2 + y^2 = z^2$.

FORMULA

The general formula for Pythagorean Triples is $x = 2st$, $y = s^2 - t^2$ and $z = s^2 + t^2$ where the conditions on s and t are that they are:

(i) s and t are natural numbers

(ii) s is greater than t

(iii) s and t have no common factors

(iv) One of s and t is even, the other is odd.

EXAMPLE

Let's show algebraically that the formula works for all values of s and t.

GroupWork

Use the formula to write down Pythagorean triples under these conditions

a) s is odd and t is even (s and $t \geq 10$)

b) s is even and t is odd (s and $t \geq 10$)

c) Either s or t is a triple digit number (s OR $t \geq 100$)

Joseph-Louis Lagrange (1736-1813) was a French mathematician who made very important contributions to various fields of mathematics, including number theory, analysis and others. In Physics, his contributions to the study of classical mechanics and the motions of the planets are well-known and celebrated.

THEOREM: **Lagrange's Four Square Theorem**

Every positive integer can be represented as the sum of the squares of four integers.

In other words, if p is any integer, there exist integers w, x y and z such that

$$p = w^2 + x^2 + y^2 + z^2$$

EXAMPLE

Show that the number 27 can be represented by the sum of four squares.

GroupWork

Show that the number 278 can be represented by the sum of four squares.