sents any one of a set of numbers; if two variables x and y are so related that whenever a value is assigned to x there is automatically assigned, by some rule or correspondence, a value to y, then we say y is a (single-valued) **function** of x. The variable x, to which values are assigned at will, is called the **independent variable,** and the variable y, whose values depend upon those of x, is called the **dependent variable.** The permissible values that x may assume constitute the **domain of definition** of the function, and the values taken on by y constitute the **range of values** of the function.

The student of mathematics used to meet the Dirichlet definition of function in his introductory course in calculus. The definition is a very broad one and does not imply anything regarding the possibility of expressing the relationship between x and y by some kind of analytic expression; it stresses the basic idea of a relationship between two sets of numbers.

Set theory has extended the concept of function to embrace relationships between any two sets of elements, be the elements numbers or anything else. Thus, in set theory, a **function** f is defined to be any set of ordered pairs of elements such that if $(a_1, b_1) \in f$, $(a_2, b_2) \in f$, and $a_1 = a_2$, then $b_1 = b_2$. The set A of all first elements of the ordered pairs is called the **domain (of definition)** of the function, and the set B of all second elements of the ordered pairs is called the **range (of values)** of the function. A functional relationship is thus nothing but a special kind of subset of the Cartesian product set $A \times B$. A **one-to-one correspondence** is, in its turn, a special kind of function, namely, a function f such that if $(a_1, b_1) \in f$, $(a_2, b_2) \in f$, and $b_1 = b_2$, then $a_1 = a_2$. If, for a functional relationship f, $(a,b) \in f$, we write $b = f(a)$.

The notion of function pervades much of mathematics, and since the early part of the present century, various influential mathematicians have advocated the employment of this concept as the unifying and central principle in the organization of elementary mathematics courses. The concept seems to form a natural and effective guide for the selection and development of textual material. There is no doubt of the value of a mathematics student's early acquaintance with the function concept.

15–4 Transfinite Numbers

The modern mathematical theory of sets is one of the most remarkable creations of the human mind. Because of the unusual boldness of some of the ideas found in its study, and because of some of the singular methods of proof to which it has given rise, the theory of sets is indescribably fascinating. Above this, the theory has assumed tremendous importance for almost the whole of mathematics. It has enormously enriched, clarified, extended, and generalized many domains of mathematics, and its role in the study of the foundations of mathematics is very basic. It also forms one of the connecting links between mathematics on the one hand and philosophy and logic on the other.

Two sets are said to be **equivalent** if and only if they can be placed in one-to-one correspondence. Two sets that are equivalent are said to have the same

cardinal number. The cardinal numbers of finite sets may be identified with the natural numbers. The cardinal numbers of infinite sets are known as **transfinite numbers,** and their theory was first developed by Georg Cantor in a remarkable series of articles beginning in 1874, and published, for the most part, in the German mathematics journals *Mathematische Annalen* and *Journal für Mathematik.* Prior to Cantor's study mathematicians accepted only one infinity, denoted by some symbol like ∞, and this symbol was employed indiscriminately to indicate the "number" of elements in such sets as the set of all natural numbers and the set of all real numbers. With Cantor's work, a whole new outlook was introduced, and a scale and arithmetic of infinities was achieved.

The basic principle that equivalent sets are to bear the same cardinal number presents us with many interesting and intriguing situations when the sets under consideration are infinite sets. Galileo Galilei observed as early as the latter part of the sixteenth century that, by the correspondence $n \leftrightarrow 2n$, the set of all positive integers can be placed in one-to-one correspondence with the set of all even positive integers. Hence, the same cardinal number should be assigned to each of these sets, and, from this point of view, we must say that there are as many even positive integers as there are positive integers in all. It is observed at once that the Euclidean postulate that states that the whole is greater than a part cannot be tolerated when cardinal numbers of infinite sets are under consideration. In fact, Dedekind, in about 1888, actually *defined* an **infinite set** to be one that is equivalent to some proper subset of itself.

We shall designate the cardinal number of the set of all natural numbers by d and describe any set having this cardinal number as being **denumerable.**[3] It follows that a set S is denumerable if and only if its elements can be written as an unending sequence $\{s_1, s_2, s_3, \ldots\}$. Since it is easily shown that any infinite set contains a denumerable subset, it follows that d is the "smallest" transfinite number.

Cantor, in one of his earliest papers on set theory, proved the denumerability of two important sets that scarcely seem at first glance to possess this property.

The first set is the set of all rational numbers. This set has the important property of being **dense.** By this is meant that between any two distinct rational numbers there exists another rational number—in fact, infinitely many other rational numbers. For example, between 0 and 1 lie the rational numbers

$$1/2,\ 2/3,\ 3/4,\ 4/5,\ 5/6,\ \ldots,\ n/(n+1),\ \ldots;$$

between 0 and 1/2 lie the rational numbers

$$1/3,\ 2/5,\ 3/7,\ 4/9,\ 5/11,\ \ldots,\ n/(2n+1),\ \ldots;$$

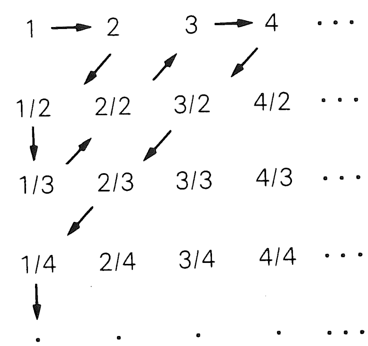
[3] Cantor designated the cardinal number by the Hebrew letter aleph with the subscript zero, that is, by $\aleph_0$.

between 0 and 1/4 lie the rational numbers

$$1/5,\ 2/9,\ 3/13,\ 4/17,\ 5/21,\ \ldots,\ n/(4n+1),\ \ldots;$$

and so on. Because of this property, one might well expect the transfinite number of the set of all rational numbers to be greater than d.[4] Cantor showed that this is *not* the case, and that, on the contrary, the set of all rational numbers is denumerable. His proof is interesting and runs as follows.

THEOREM 1: *The set of all rational numbers is denumerable.*
Consider the array



in which the first row contains, in order of magnitude, all the natural numbers (that is, all positive fractions with denominator 1), the second row contains, in order of magnitude, all the positive fractions with denominator 2, the third row contains, in order of magnitude, all the positive fractions with denominator 3, etc. Obviously, every positive rational number appears in this array, and if we list the numbers in the order of succession indicated by the arrows, omitting numbers that have already appeared, we obtain an unending sequence

$$1,\ 2,\ 1/2,\ 1/3,\ 3,\ 4,\ 3/2,\ 2/3,\ 1/4,\ \ldots$$

in which each positive rational number appears once and only once. Denote this sequence by $\{r_1, r_2, r_3, \ldots\}$. Then the sequence $\{0, -r_1, r_1, -r_2, r_2, \ldots\}$ contains the set of all rational numbers, and the denumerability of this set is established.

The second set considered by Cantor is a seemingly much more extensive set of numbers than the set of rational numbers. We first make the following definition.

DEFINITION 1: A complex number is said to be **algebraic** if it is a root of some polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

[4] The cardinal number of a set A is said to be *greater than* the cardinal number of a set B if and only if B is equivalent to a proper subset of A, but A is equivalent to no proper subset of B.

where $a_0 \neq 0$ and all the a_k's are integers. A complex number that is not algebraic is said to be **transcendental.**

It is quite clear that the algebraic numbers include, among others, all rational numbers and all roots of such numbers. Accordingly, the following theorem is somewhat astonishing:

THEOREM 2: *The set of all algebraic numbers is denumerable.*

Let $f(x)$ be a polynomial of the kind described in Definition 1, where, without loss of generality, we may suppose $a_0 > 0$. Consider the so-called **height** of the polynomial, defined by

$$h = n + a_0 + |a_1| + |a_2| + \cdots + |a_{n-1}| + |a_n|.$$

Obviously h is an integer $\geqq 1$, and there are plainly only a finite number of polynomials of a given height h, and therefore only a finite number of algebraic numbers arising from polynomials of a given height h. We may now list (theoretically speaking) all the algebraic numbers, refraining from repeating any number already listed, by first taking those arising from polynomials of height 1, then those arising from polynomials of height 2, then those arising from polynomials of height 3, and so on. We thus see that the set of all algebraic numbers can be listed in an unending sequence, whence the set is denumerable.

In view of the preceding two theorems, there remains the possibility that all infinite sets are denumerable. That this is not so was shown by Cantor in a striking proof of the following significant theorem:

THEOREM 3: *The set of all real numbers in the interval* $0 < x < 1$ *is nondenumerable.*

The proof is indirect and employs an unusual method known as the **Cantor diagonal process.** Let us, then, assume the set to be denumerable. Then we may list the numbers of the set in a sequence $\{p_1, p_2, p_3, \ldots\}$. Each of these numbers p_i can be written uniquely as a nonterminating decimal fraction; in this connection, it is useful to recall that every rational number may be written as a "repeating decimal"; a number such as 0.3, for example, can be written as 0.29999. . . . We can then display the sequence in the following array,

$$p_1 = 0.a_{11}a_{12}a_{13}\cdots$$
$$p_2 = 0.a_{21}a_{22}a_{23}\cdots$$
$$p_3 = 0.a_{31}a_{32}a_{33}\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$

where each symbol a_{ij} represents some one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Now, in spite of any care that has been taken to list all the real numbers between 0 and 1, there is a number that could not have been listed. Such a number is $0.b_1b_2b_3\ldots$, where, say, $b_k = 7$ if $a_{kk} \neq 7$ and $b_k = 3$ if $a_{kk} = 7$, for $k = 1, 2, 3, \ldots, n, \ldots$. This number clearly lies between 0 and 1, and it must differ from each number p_i, for it differs from p_1 in at least the first decimal place, from p_2 in at least the second decimal place, from p_3 in at least the third decimal place, and so on. Thus, the original assumption that all the real numbers between 0 and 1 can be listed in a sequence is untenable, and the set must therefore be nondenumerable.

Cantor deduced the following remarkable consequence of Theorems 2 and 3:

THEOREM 4: *Transcendental numbers exist.*
Since, by Theorem 3, the set of all real numbers between 0 and 1 is nondenumerable, it is easily demonstrated that the set of all complex numbers is also nondenumerable. By Theorem 2, however, the set of all algebraic numbers is denumerable. It follows that there must exist complex numbers that are not algebraic, and the theorem is established.

Not all mathematicians are willing to accept the above proof of Theorem 4. The acceptability or nonacceptability of the proof hinges on what one believes mathematical existence to be, and there are some mathematicians who feel that mathematical existence is established only when one of the objects whose existence is in question is actually constructed and exhibited. Now the above proof does not establish the existence of transcendental numbers by producing a specific example of such a number. There are many existence proofs in mathematics of this nonconstructive sort, where existence is presumably established by merely showing that the assumption of nonexistence leads to a contradiction. Most proofs of the Fundamental Theorem of Algebra, for example, are formulated along such lines.

Because of the dissatisfaction of some mathematicians with nonconstructive existence proofs, a good deal of effort has been made to replace such proofs by those that actually yield one of the objects concerned.

The proof of the existence of transcendental numbers and the proof that some particular number is transcendental are two quite different matters, the latter often being a very difficult problem. It was Hermite who, in 1873, proved that the number e, the base for natural logarithms, is transcendental, and Lindemann, in 1882, who first established the transcendentality of the number π. Unfortunately, it is inconvenient for us to prove these interesting facts here. The difficulty of identifying a particular given number as algebraic or transcendental is illustrated by the fact that it is not yet known whether the number π^π is algebraic or transcendental. A recent gain along these lines was the establishment of the transcendental character of any number of the form a^b, where a is an algebraic number different from 0 or 1, and b is any irrational algebraic number. This result, achieved in 1934 by Alexsander Osipovich Gelfond (1906–1968), and now known as Gelfond's theorem, was a culmination of an almost thirty-year effort to prove that the so-called **Hilbert number,** $2^{\sqrt{2}}$, is transcendental.

Since the set of all real numbers in the interval $0 < x < 1$ is nondenumerable, the transfinite number of this set is greater than d. We shall denote it by c, and shall refer to it as the **cardinal number of the continuum.** It is generally believed that c is the next transfinite number after d—that is, that there is no set having a cardinal number greater than d but less than c. This belief is known as the **continuum hypothesis,** but, in spite of strenuous efforts, no proof has been found to establish it. Many consequences of the hypothesis have been deduced. In about 1940, the Austrian logician Kurt Gödel (1906–1978) succeeded in showing that the continuum hypothesis is consistent with a famous postulate set of set theory provided these postulates themselves are consistent. Gödel conjectured that the denial of the continuum hypothesis is also consistent with the postulates of set theory. This conjecture was established, in 1963, by Paul J. Cohen (born 1934) of Stanford University, thus proving that the continuum hypothesis is independent of the postulates of set theory, and hence can never be deduced from those postulates. The situation is analogous to that of the parallel postulate in Euclidean geometry.

It has been shown that the set of all single-valued functions $f(x)$ defined over the interval $0 < x < 1$ has a cardinal number greater than c, but whether this cardinal number is the next after c is not known. Cantor's theory provides for an infinite sequence of transfinite numbers, and there are demonstrations that purport to show that an unlimited number of cardinal numbers greater than that of the continuum actually exist.

15–5 Topology

Topology started as a branch of geometry, but during the second quarter of the twentieth century it underwent such generalization and became involved with so many other branches of mathematics that it is now perhaps more properly considered, along with geometry, algebra, and analysis, as a fundamental division of mathematics. Today, topology may be roughly defined as the mathematical study of continuity. In this section we shall restrict ourselves to some of those aspects of the subject that reflect its geometric origin. From this point of view, topology may be regarded as the study of those properties of geometric figures that remain invariant under so-called **topological transformations;** that is, under single-valued continuous mappings possessing single-valued continuous inverses. By a geometric figure, we mean a point set in three-dimensional (or higher-dimensional) space; a single-valued continuous mapping is one that, given a Cartesian coordinate system in the space, can be represented by single-valued continuous functions of the coordinates.

Since the set of all topological transformations of a geometric figure constitute a transformation group, topology can, from our viewpoint, be considered as a Kleinian geometry, and hence codified within Klein's *Erlanger Programm.* Those properties of a geometric figure that remain invariant under topological transformations of the figure are called **topological properties** of the figure, and two figures that can be topologically transformed into one another are said to be **homeomorphic,** or **topologically equivalent.**

The mapping functions of a topological transformation need not be defined over the whole of the space in which the geometric figure is imbedded, but may