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AN INTRODUCTION TO THE HISTORY OF MATHEMATICS

with Cultural Connections
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Sixth Edition

THE SAUNDERS SERIES

The Pythagorean philosophy rested on the assumption that whole number is the cause of the various qualities of man and matter. This led to an exaltation and study of number properties, and arithmetic (considered as the theory of numbers), along with geometry, music, and spherics (astronomy), constituted the fundamental liberal arts of the Pythagorean program of study. This group of subjects became known in the Middle Ages as the **quadrivium,** to which was added the **trivium** of grammar, logic, and rhetoric. These seven liberal arts came to be looked upon as the necessary equipment of an educated person.

Because Pythagoras' teaching was entirely oral, and because of the brotherhood's custom of referring all discoveries back to the revered founder, it is now difficult to know just which mathematical findings should be credited to Pythagoras himself and which to other members of the fraternity.

3–3 Pythagorean Arithmetic

The ancient Greeks made a distinction between the study of the abstract relationships connecting numbers and the practical art of computing with numbers. The former was known as **arithmetic** and the latter as **logistic.** This classification persisted through the Middle Ages until about the close of the fifteenth century, when texts appeared treating both the theoretical and practical aspects of number work under the single name *arithmetic*. It is interesting that today *arithmetic* has its original significance in continental Europe, while in England and America the popular meaning of *arithmetic* is synonymous with that of ancient *logistic*. In these two countries, the descriptive term **number theory** is used to denote the abstract side of number study.

It is generally conceded that Pythagoras and his followers, in conjunction with the fraternity's philosophy, took the first steps in the development of number theory, and at the same time laid much of the basis of future number mysticism. Thus, Iamblichus, an influential Neoplatonic philosopher of about A.D. 320, has ascribed to Pythagoras the discovery of **amicable,** or **friendly, numbers.** Two numbers are amicable if each is the sum of the proper divisors[3] of the other. For example, 284 and 220, constituting the pair ascribed to Pythagoras, are amicable, since the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110, and the sum of these is 284, whereas the proper divisors of 284 are 1, 2, 4, 71, 142, and the sum of these is 220. This pair of numbers attained a mystical aura, and superstition later maintained that two talismans bearing these numbers would seal perfect friendship between the wearers. The numbers came to play an important role in magic, sorcery, astrology, and the casting of horoscopes. It seemed that no new pair of amicable numbers was discovered until the great French number theorist Pierre de Fermat in 1636 announced 17,296 and 18,416 as another pair. It has recently been established,

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[3] The **proper divisors** of a positive integer N are all the positive integral divisors of N except N itself. Note that 1 is a proper divisor of N. A somewhat antiquated synonym for proper divisor is **aliquot part.**

however, that this was a rediscovery, and that this pair of amicable numbers had been previously found by the Arab al-Banna (1256–1321) in the late thirteenth or early fourteenth century, perhaps by using the Ṯâbit ibn Qorra formula. (For this formula, see Problem Study 7.11.) Two years after Fermat's announcement, the French mathematician and philosopher René Descartes gave a third pair. The Swiss mathematician Leonhard Euler undertook a systematic search for amicable numbers and, in 1747, gave a list of thirty pairs, which he later extended to more than sixty. A curiosity in the history of these numbers was the late discovery, by the sixteen-year-old Italian boy Nicolo Paganini[4] in 1866, of the overlooked and relatively small pair of amicable numbers, 1184 and 1210. All amicable number pairs below one billion have now been found.

Other numbers having mystical connections essential to numerological speculations, and sometimes ascribed to the Pythagoreans, are the **perfect, deficient,** and **abundant numbers.** A number is *perfect* if it is the sum of its proper divisors, *deficient* if it exceeds the sum of its proper divisors, and *abundant* if it is less than the sum of its proper divisors. So God created the world in six days, a perfect number, since $6 = 1 + 2 + 3$. On the other hand, as Alcuin (735–804) observed, the whole human race descended from the eight souls of Noah's ark, and this second creation was imperfect, for 8, being greater than $1 + 2 + 4$, is deficient. Until 1952, there were only twelve known perfect numbers, all of them even numbers, of which the first three are 6, 28, and 496. The last proposition of the ninth book of Euclid's *Elements* (ca. 300 B.C.) proves that *if* $2^n - 1$ *is a prime number,*[5] *then* $2^{n-1}(2^n - 1)$ *is a perfect number.* The perfect numbers given by Euclid's formula are even numbers, and Euler has shown that every even perfect number must be of this form. The existence or nonexistence of odd perfect numbers is one of the celebrated unsolved problems in number theory. There certainly is no number of this type having less than 200 digits.

In 1952, with the aid of the SWAC digital computer, five more perfect numbers were discovered, corresponding to $n = 521, 607, 1279, 2203$, and 2281 in Euclid's formula. In 1957, using the Swedish machine BESK, another was found, corresponding to $n = 3217$. In 1961, with an IBM 7090, two more were found, for $n = 4253$ and 4423. There are no other even perfect numbers for $n < 5000$. The values $n = 9689, 9941, 11213, 19937, 21701, 23209, 86243, 132049$, and 216091 also yield perfect numbers, bringing the list of known perfect numbers to thirty. The last was found by scientists at Chevron in 1985 on a $10,000,000 Cray X-MP supercomputer.

The concept of perfect numbers has inspired certain generalizations by modern mathematicians. If we let $\sigma(n)$ represent the sum of *all* the divisors of n

[4] Not to be confused with Nicolo Paganini (1782–1840), the noted Italian violinist and composer.

[5] A **prime number** is a positive integer greater than 1 and having no positive integral divisors other than itself and unity. An integer greater than 1 that is not a prime number is called a **composite number,** thus, 7 is a prime number, whereas 12 is a composite number.

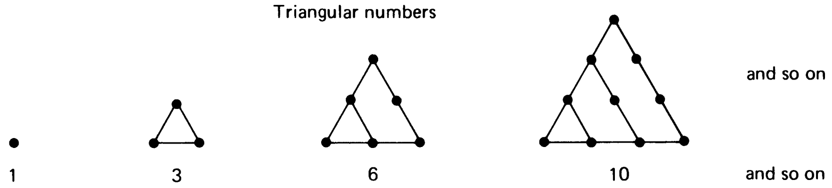

FIGURE 7

(including n itself), then n is perfect if and only if $\sigma(n) = 2n$. In general, if we should have $\sigma(n) = kn$, where k is a natural number, then n is said to be **k-tuply perfect.** One can show, for example, that 120 and 672 are triply perfect. It is not known if infinitely many multiply perfect numbers, let alone just perfect ones, exist. It is also not known if any odd multiply perfect number exists. In 1944, the concept of **superabundant numbers** was created. A natural number n is *superabundant* if and only if $\sigma(n)/n > \sigma(k)/k$ for all $k < n$. It is known that there are infinitely many superabundant numbers. Other numbers related to perfect, deficient, and abundant numbers that have been introduced in recent times are *practical numbers, quasiperfect numbers, semiperfect numbers,* and *weird numbers.* We merely mention these concepts to illustrate how ancient number work has inspired related modern investigations.

Although not all historians of mathematics feel that amicable and perfect numbers can be ascribed to the Pythagoreans, there seems to be universal agreement that the **figurate numbers** did originate with the earliest members of the society. These numbers, considered as the number of dots in certain geometrical configurations, represent a link between geometry and arithmetic. Figures 7, 8, and 9 account for the geometrical nomenclature of **triangular numbers, square numbers, pentagonal numbers,** and so on.

Many interesting theorems concerning figurate numbers can be established in purely geometric fashion. To show Theorem I (*any square number is the sum of two successive triangular numbers*), for example, we observe that a square number, in its geometric form, can be divided as in Figure 10. Again, Figure 11 illustrates Theorem II (*the nth pentagonal number is equal to n plus three times the* ($n - 1$)*th triangular number*). Theorem III (*the sum of any number of consecutive odd integers, starting with* 1, *is a perfect square*) is exhibited geometrically by Figure 12.

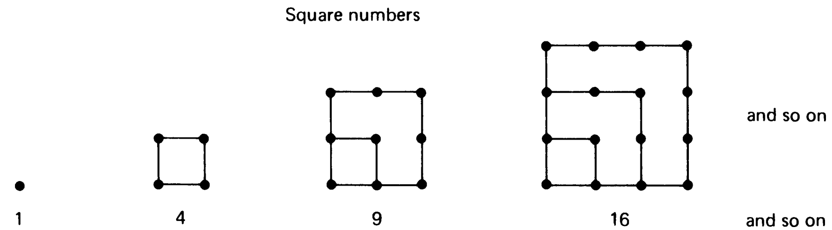

FIGURE 8

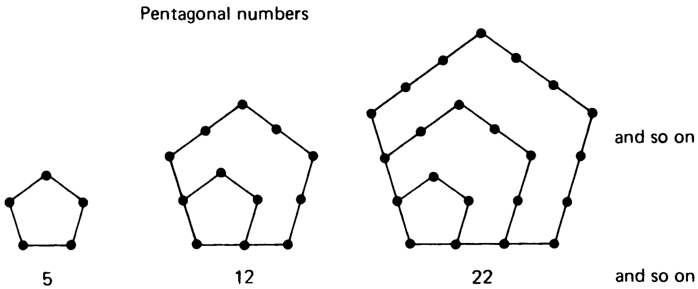

FIGURE 9

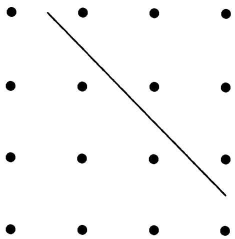

FIGURE 10

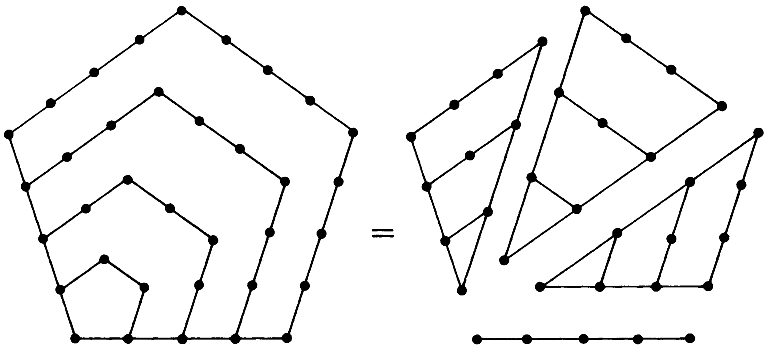

FIGURE 11

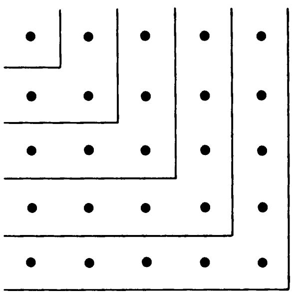

FIGURE 12